The Drunk Bird and the Mathematics of No Return

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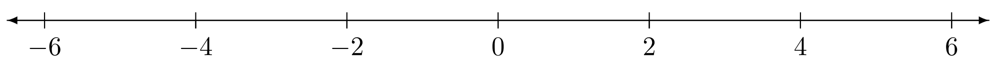
1 Introduction

This Essay is about the famous "Random Walk" problem and its extension into further dimensions. An introduction to the problem is as follows. Imagine a walker on a number line who starts at the origin and flips an unbiased coin where heads means he takes a positive step and tails means he takes a negative step. In 2D the walker walks on a plane and has 4 directions to choose from and in 3D the walker travels in a lattice giving 6 options. Already you may have noticed that the probability of choosing one direction over another is thus $\frac{1}{2d}$ where d is the number of dimensions (providing unbiased choice of course). However the question that interested Hungarian mathematician George Pólya and the question that will be the focus of this essay is if it is guaranteed for a walker to return to the origin given an infinite amount of time.

2 The 1D case



For the simple One dimensional case the walker starts at the origin of a number line, and can move in the positive or negative direction by 1 unit.The walker continues to walk for an infinite amount of time. What is the probability that the walker will return to the origin? Of course it is one and this makes logical sense with infinite time the walker will cover every point an infinite number of times. However a proof is more sufficient as to avoid misjudgment. There exist 2^{2n} distinct trajectories of length 2n since there are always two possible choices. However the number of positive and negative steps must be equal to return to the origin. Thus the number of possible trajectories is $\binom{2n}{n}$and therefore the probability of returning to the origin after 2n steps is:

$$\frac{\binom{2n}{n}}{2^{2n}} = \frac{\frac{(2n)!}{n!(2n-n)!}}{2^{2n}} = \frac{(2n)!}{(n!)^2 2^{2n}}$$

Which can be approximated with Stirling's approximation as:

$$\sim \frac{1}{\sqrt{\pi n}}$$

thus the expected number of returns =

$$\frac{1}{\sqrt{\pi}} \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$$

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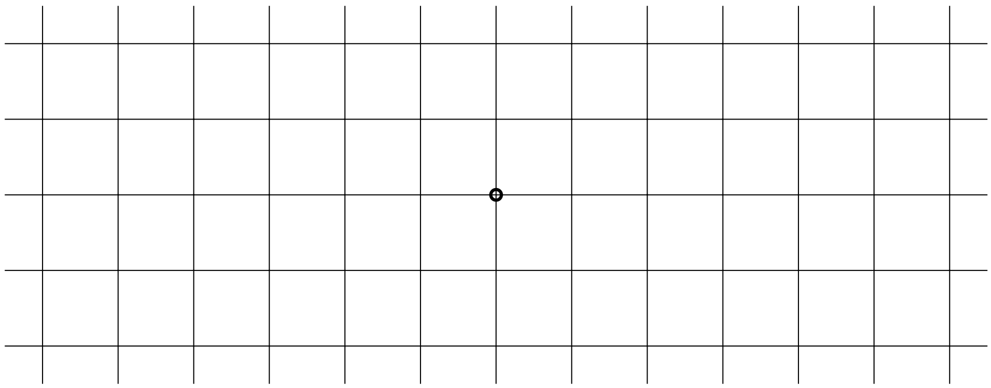
3 The 2D case



The 2D case is analogous to two 1D cases multiplied by a constant since each walker only walks half the number of times due to the fact that the overall walker only walks 1 unit, therefore the probability for the point to be at the origin = the probability of two 1D walkers being at the origin

$$= C(\frac{1}{\sqrt{\pi n}})^2 = \frac{C}{\pi n}$$

Thus the expected number of returns =

$$\frac{C}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}$$

which is a multiple of the well known diverging harmonic series.The proof for the divergence of the harmonic series is as follows:

$$\text{let} \quad S_1 = \sum_{n=1}^{\infty}\frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + ...$$

now examine a similar series:

$$S_2 = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{4} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + \frac{1}{8} + ...$$

S_2 is trivially less than S_1 and since S_2 can be simplified as follows:

$$S_2 = 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + ...$$

The addition of a constant term means that the series diverges, and therefore so does S_1. This also explains why

$$\frac{1}{\sqrt{\pi}}\sum_{n=1}^{\infty}\frac{1}{n}$$

also diverges since this sum (say S_3) is even greater as each term is divided by a smaller value (the square root). Since the sums diverge the number of returns to the origin is ∞ which means that the origin is recurrent implying every other point is also recurrent.

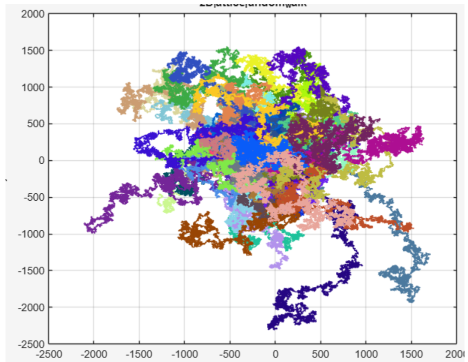


Figure 1: 2D random walk simulation on matlab

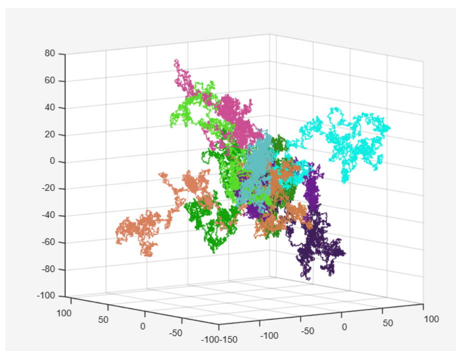


Figure 2: 3D random walk simulation on matlab

4 The 3D case

Just like in the 2D case, the 3D case is analogous to two 3 1D walkers multiplied by a constant and thus the probability of return =

$$C(\frac{1}{\sqrt{\pi n}})^3 = \frac{C}{\pi^{3/2} n^{3/2}}$$

and thus the number of expected returns

$$= \frac{C}{\pi^{3/2}} \sum_{1}^{\infty} \frac{1}{n^{3/2}}$$

however unlike the previous examples the power of the p series > 1 and therefore it converges which means that the origin is not recurrent and the walker is not guaranteed to return.

5 Building Intuition

To build an intuition on this extraordinary result we must consider the expected number of steps traveled when you have reached a distance r. First let:

$$S_n = \sum_{i=1}^{n} X_i$$

where X_i are the individual steps (+1 or -1) and S_n is the sum of the steps and therefore the location at step n. now lets define some expected values:

$$E[X_i] = 0$$

$$E[X_i^2] = 1$$

$$E[S_n^2] = E[(X_1+X_2+X_3+...)(X_1+X_2+X_3+...)] = E[\sum_{i=1}^{n} X_i^2 + \sum_{i \neq j}^{n} X_i X_j] = \sum_{i=1}^{n} 1 = n$$

thus the expected distance traveled after n steps is $\sqrt{n}$ so for a distance of n the expected number of steps is n^2. However you may inquire that this is a derivation only for the 1D case what about for 3 dimensions? well for 3 dimensions S_n is represented by a vector of 3 components (x,y,z) and by Pythagoras:

$$(S_n)^2 = (S_n^x)^2 + (S_n^y)^2 + (S_n^z)^2$$
$$= d(\frac{n}{d})$$

by applying the 1D argument for each dimension. Now if we consider the total number of points within distance n, it will be proportional to

$$n^3$$

so as time goes on the walk will cover an ever decreasing fraction of these points proportional to:

$$\frac{r^2}{r^3}$$

In other words the possible number of occupiable points grows faster than the rate at which points are occupied and therefore no point is recurrent including the origin. This method works for all dimensions d. The pattern also continues for higher dimensions and a walker is not guaranteed to return to the origin if d > 2.

6 Conclusion

In conclusion this is a beautiful and simultaneously mind bending piece of mathematics that was really fun to write an essay on and given me even more excitement for the degree in maths I hope to pursue. Thank you so much Tom for running this competition. And thank you for reading.

”A drunk man will find his way home, but a drunk bird may get lost forever” - Shizuo Kakutani .