The Einstein Equation

By Veer Parekh

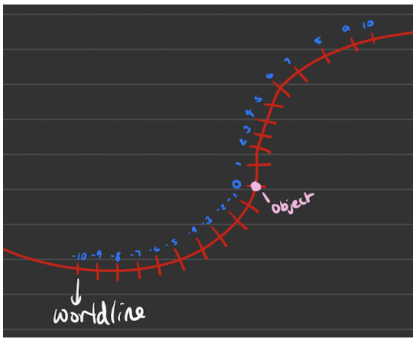
April 2026

Introduction

The Einstein equation (not $E = mc^2$) is a very famous equation and it explains how spacetime curves, depending on the energy content in the universe. But how do you show that mathematically?

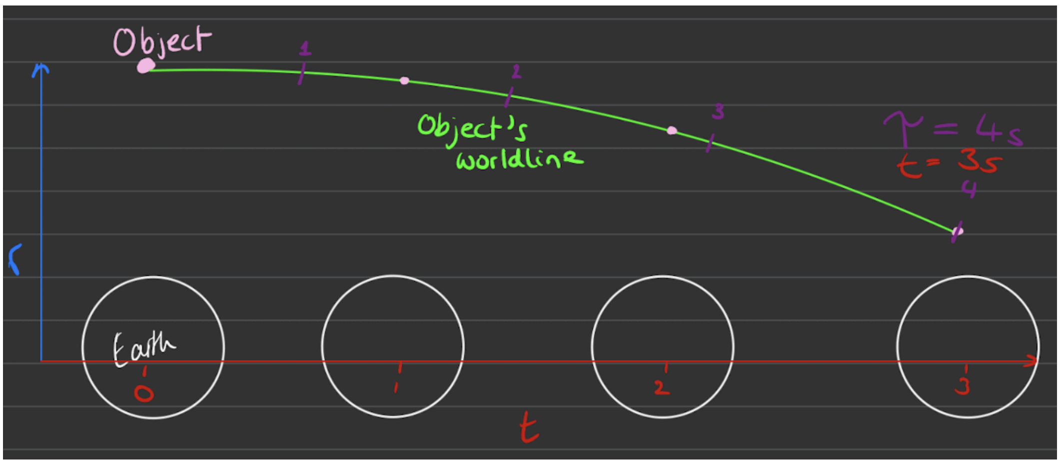
Spacetime and Worldlines

Spacetime is a concept that links 3D space and 1D time into a 4D spacetime, and a worldline describes the motion of an object through this spacetime.



Here is an object's worldline in space-time. It is split into graduations which each represent that object's proper time (τ). Proper time is the time that governs the object's internal evolution.

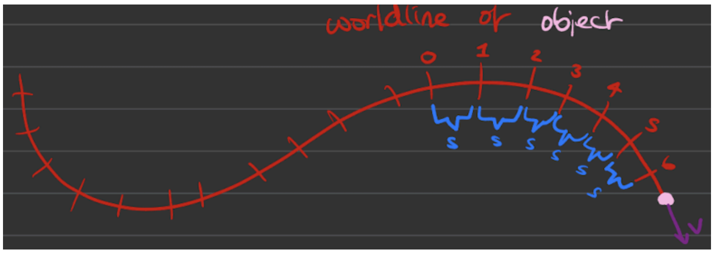
As proper time goes by, a coordinate system is also needed to describe the position of an object.



In the diagram above, we have an object falling to Earth. This coordinate system uses time(*t*) and distance from Earth(*r*). However it is important to note that the time here is the time of an outside observer and is different from the proper time. In this example, the proper time passed is 4 seconds while time is 3 seconds.

Spacetime velocity

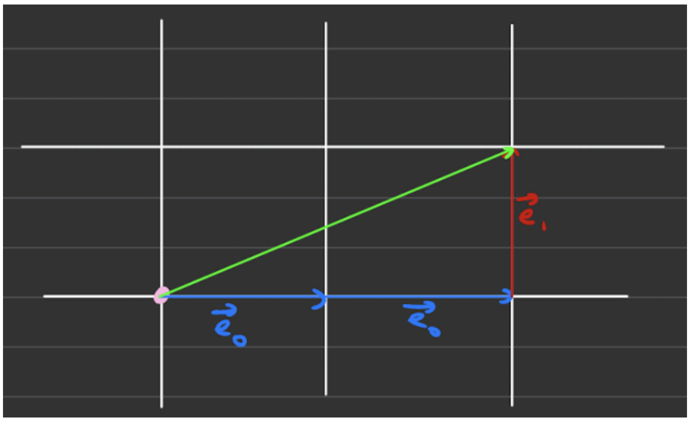
Nothing moves stationary in spacetime as it always moves in time and it just distributes its motion in space and time separately. This means that across a worldline, in between each interval of proper time, there is the same distance (*s*).



This means the velocity of the object only depends on the direction as the speed through spacetime is the same. This speed is the speed of light (*c*).

We can write our first equation: $\left|\left|\vec{v}\right|\right| = c$

The velocity vector can also be written as a sum of its components



For example, this vector can be written as:

$\vec{v} = 2\vec{e}_0 + 1\vec{e}_1$

A more general equation:

$\vec{v} = v^0\vec{e}_0 + v^1\vec{e}_1$

Einstein has his notation for this:

$\vec{v} = v^\alpha\vec{e}_\alpha$

The numbers are replaced as greek letters and when a greek letter is repeated twice, once as superscript and once as subscript, it means to take the sum of.

Geodesics

When no force is applied, worldlines are straight. This means we can predict the velocity of an object just by transporting the velocity along itself to tell us the object's trajectory. This trajectory of an object is called the geodesic. Every object travels along a geodesic and because the vector doesn't change along a geodesic, we can say:

$\frac{d\vec{v}}{d\tau} = \vec{0}$

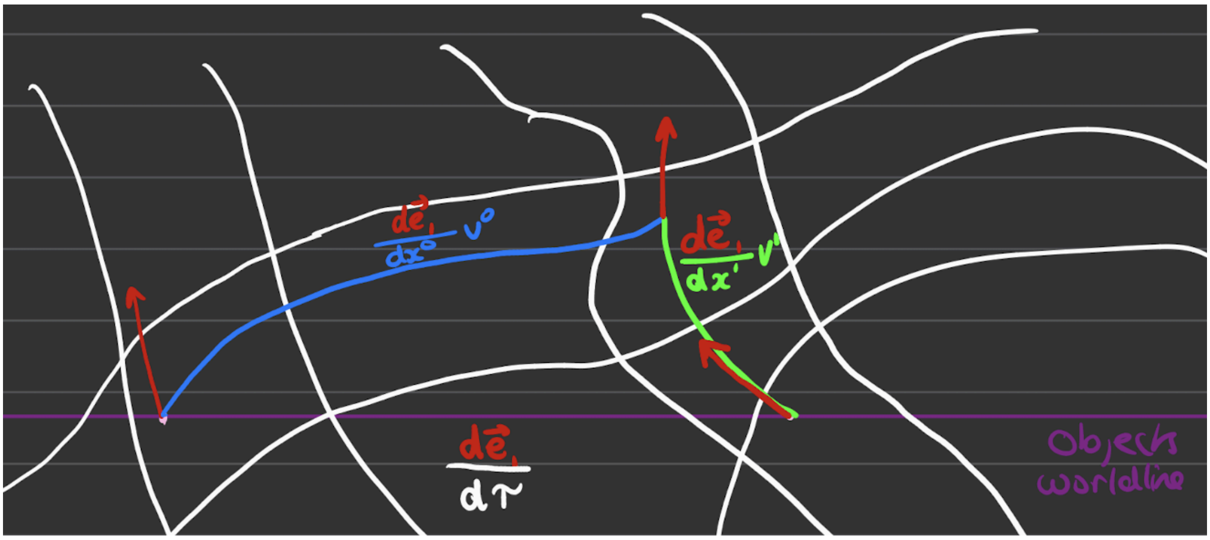
Earlier we expressed v as its components, so if we replace it here we get:

$\frac{d(v^\alpha\vec{e}_\alpha)}{d\tau} = \vec{0}$

Using the product rule, we can rewrite this as:

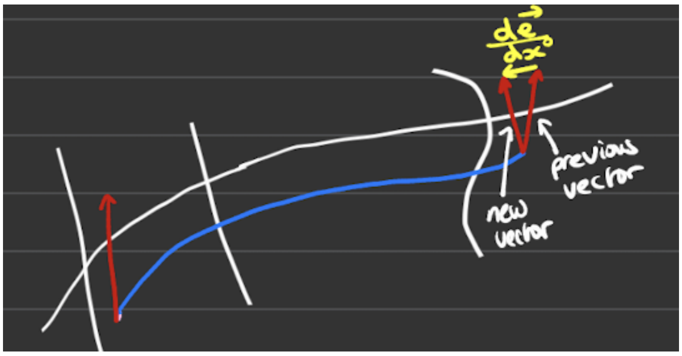
$$v^{\alpha}\frac{d\vec{e}_{\alpha}}{d\tau} + \frac{dv^{\alpha}}{d\tau}\vec{e}_{\alpha} = \vec{0}$$

The first derivative here talks about the change in basis vectors themselves. This could happen because the grid can be irregular. This means the components on the grid could vary even if the vector is the same.

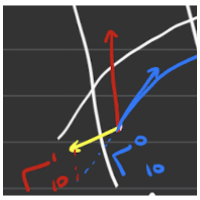


Here is a very irregular grid. The red lines show how a basis vector changes because the grid is irregular. The evolution of the basis vector along a worldline can be decomposed as the sum of its evolution along each of its two components, multiplied by the speed of the object, because the faster it moves, the faster the basis vector will vary. This is shown above by the blue and green lines.

For each coordinate we can see how each basis vector varies.



We can see that we can express the change in the two vectors as a new vector in yellow. This new vector no longer depends on the trajectory but only on the structure of the grid itself.



This vector can then be split into its components, which is shown with the capital gamma (Γ). These are called Christoffel symbols, and there are 8 of them in a 2D case, 2 components for 4 different vectors. They tell you how exactly the grid changes.
We can simplify all of that into this equation:

$$\frac{d\vec{e}_{\alpha}}{d\tau} = \Gamma^{\gamma}{}_{\alpha\beta}v^{\beta}\vec{e}_{\gamma}$$

Using the previous equation:

$$v^{\alpha}\frac{d\vec{e}_{\alpha}}{d\tau} + \frac{dv^{\alpha}}{d\tau}\vec{e}_{\alpha} = \vec{0}$$

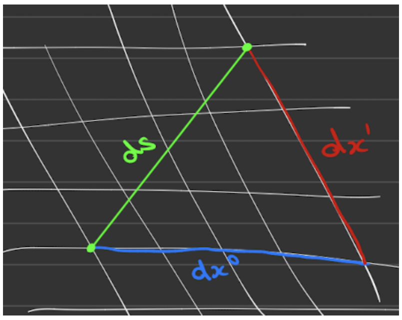
We can substitute in the first one into the second one and rearrange:

$$\frac{dv^{\alpha}}{d\tau} = -\Gamma^{\alpha}_{\mu\nu}v^{\mu}v^{\nu}$$

This equation tells us the rate of change of each component of the velocity as proper time passes. You can therefore predict the whole trajectory of an object as long as you know its velocity at a moment and the value of each Chrisoffel symbol throughout the grid.

The Metric Tensor

Now we are able to predict the movement of objects as long as we know the Christoffel symbols. However, we don't know the distances between two points on an irregular grid.



In this diagram we have a small distance ds. This looks like we could just use Pythagoras' theorem but we can't because the intervals on an irregular grid don't have the same distance apart. To find the distance, we have to use this formula:

$$(ds)^2 = a\,dx^0\,dx^0 + b\,dx^0\,dx^1 + c\,dx^1\,dx^0 + d\,dx^1\,dx^1$$

The constants (a,b,c,d) depend on the shape of the grid. When the grid is regular, a = 1, b = 0, c = 0 and d = 1. This leads back to Pythagoras' theorem. These constants can then be written in a table:

	dx^0	dx^1
dx^0	a	b
dx^1	c	d

This table is known as the metric tensor and is denoted with the letter g and is written without the headings dx^0 and dx^1:

$$g = \begin{array}{|c|c|} \hline g_{00} & g_{01} \\ \hline g_{10} & g_{11} \\ \hline \end{array}$$

From here we can write an equation for our small distance using Einstein notation:

$$(ds)^2 = g_{\mu\nu} dx^{\mu} dx^{\nu}$$

We can also choose to replace this distance by a velocity vector and replace the coordinate differences by the components of the vector:

$$\left|\left|\vec{v}\right|\right| = g_{\mu\nu} v^{\mu} v^{\nu}$$

Using the first equation we formed we can therefore say:

$$c^2 = g_{\mu\nu} v^{\mu} v^{\nu}$$

The metric tensor allows us to calculate the Christoffel symbols. The Christoffel symbols tell us how basis vectors vary along the grid, and these basis vectors are related through the shape of the grid. The shape of the grid is expressed by the metric tensor itself. By measuring how the metric tensor varies along the grid, we can determine how the basis vectors change and allow us to calculate the Christoffel symbols. The equation for this is the following:

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{g^{\gamma\sigma}}{2}\left(\frac{dg_{\sigma\alpha}}{dx^{\beta}} + \frac{dg_{\sigma\beta}}{dx^{\alpha}} - \frac{dg_{\alpha\beta}}{dx^{\sigma}}\right)$$

This equation includes derivatives of the metric tensor, as we see how it varies along the grid, and the inverse of the metric tensor, denoted with the indices at the top. This inverse metric is hard to calculate so sometimes we can simplify the equation. For example, if the two axes are always perpendicular, then we can use this equation:

$$\Gamma^{\gamma}{}_{\alpha\beta} = \frac{1}{2g_{\gamma\gamma}}\left(\frac{dg_{\sigma\alpha}}{dx^{\beta}} + \frac{dg_{\sigma\beta}}{dx^{\alpha}} - \frac{dg_{\alpha\beta}}{dx^{\sigma}}\right)$$

Now we can predict the trajectory of an object, as long as we know the metric tensor, using the geodesic equation.

We can use what we have so far to predict time dilation. In empty spacetime, the metric used is called the Minkowski metric. It is the same everywhere as long as you're describing empty spacetime. The metric looks like this:

$$g = \begin{array}{c|c|c} & dt & dx \\ \hline dt & c^2 & 0 \\ \hline dx & 0 & -1 \end{array}$$

Because this metric is the same everywhere, this means the derivatives are all 0. This means the Christoffel symbols are also 0 and therefore, using the geodesic equation:

$$\frac{dv^\alpha}{d\tau} = 0$$

This means an object going through this empty spacetime will go straight. To predict time dilation, we use the equation from earlier and substitute in the Minkowski metric.

$$c^2 = g_{\mu\nu} v^\mu v^\nu$$

$$c^2 = c^2\, v^t v^t + 0v^t v^x + 0v^x v^t - 1v^x v^x$$

$$c^2 = c^2 (v^t)^2 - (v^x)^2$$

$$1 = (v^t)^2 - \frac{(v^x)^2}{c^2}$$

$$(v^t)^2 = 1 + \frac{(v^x)^2}{c^2}$$

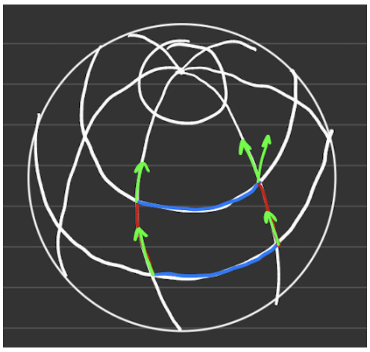
$$v^t = \sqrt{1 + \frac{(v^x)^2}{c^2}}$$

This equation tells you that as v^x increases (the speed of the object) the faster our time passes compared to the object's proper time. This is time dilation.

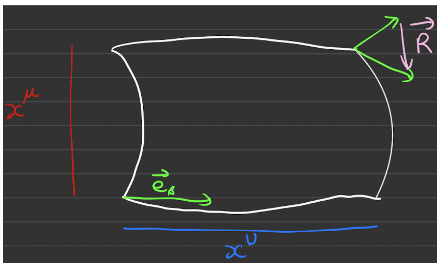
Curvature

Spacetime isn't always flat. The curvature of spacetime is more commonly known as gravity.

Here is a flat plane. If we were to transport the bottom left green vector in any direction, it would always end up in the same direction at the top right.



Here is a curved plane. The same vector could be transported first right and then up, or first up and then right, and it would give a different vector, shown by two different green arrows at the top.



The difference of the two vectors can be shown as $\vec{R}$

If the surface is flat then $\vec{R} = 0$

We can show this using this equation:

$$\vec{R} = \frac{d}{dx^\mu}\frac{d}{dx^\nu}\vec{e}_\beta - \frac{d}{dx^\nu}\frac{d}{dx^\mu}\vec{e}_\beta$$

In this equation we have the derivative of basis vectors, which give Christoffel symbols. If we put the Christoffel symbols in, and then break $\vec{R}$ into its components, we get a very long equation.

$$R^\alpha{}_{\beta\mu\nu} = \frac{d\Gamma^\alpha{}_{\beta\nu}}{dx^\mu} + \Gamma^\lambda{}_{\beta\nu}\Gamma^\alpha{}_{\lambda\mu} - \frac{d\Gamma^\alpha{}_{\beta\mu}}{dx^\nu} - \Gamma^\lambda{}_{\beta\mu}\Gamma^\alpha{}_{\lambda\nu}$$

There are 16 different possibilities for these components and it is called the Riemann curvature tensor and fully describes the curvature of a surface. To simplify it though, we use the Ricci Tensor:

$$R_{\mu\nu} = \begin{array}{|c|c|} \hline R_{00} & R_{01} \\ \hline R_{10} & R_{11} \\ \hline \end{array}$$

To find the value of the Ricci Tensor, you choose those from the Riemann curvature tensor of which have the same second and fourth indices as the Ricci Tensor. Then out of them, you do the sum of the ones that have the same 1st and 2nd index as each other. For example:

$$R_{01} = R^{0}{}_{001} + R^{1}{}_{011}$$

A more general equation is:

$$R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$$

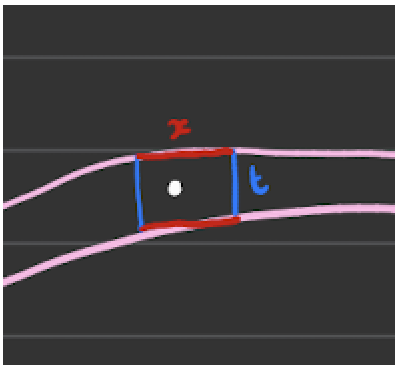
The Ricci Tensor tells you how volumes change on the surface as you move along its curvature. The average curvature in all directions can also be calculated and it's called the Ricci scalar, R.

$$R = g^{\mu\nu} R_{\mu\nu}$$

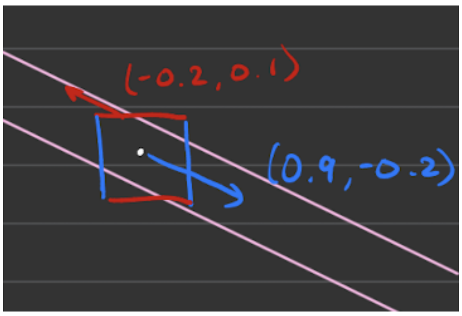
The Riemann curvature tensor, Ricci tensor, and Ricci scalar all come together to explain the curvature of a surface.

Energy Fluxes

Energy moves through spacetime like a fluid.



Here is the pink trajectory in spacetime and there are 4 sides where energy can be passed. However the situation is symmetrical, so if energy comes in one side, it must exit the other side. Therefore only two surfaces are needed to describe the fluxes of energy.



You can describe these fluxes as vectors and you can break them into their components. These components can be placed in another table, called the energy-momentum tensor and it tells you how momentum is transferred along different coordinates.

		t	x
T =	t	T_{tt}	T_{tx}
	x	T_{xt}	T_{xx}

T_{tt} - How much energy moves through time. This is energy density
T_{tx} and T_{xt} - Both are equal and tell you how much energy travels through space over time. This is momentum.
T_{xx} - How much energy is moving through space. This is the tendency energy has to move in one direction, the pressure.

In 4D spacetime, there are 16 components instead of 4 and you also have to look at viscosity, which is the tendency energy has to transfer its motion to its surroundings.

The Einstein Equation

Linking everything together, we can now explain the Einstein equation. It relates the curvature to the energy content of the universe. This equation can't be proven using our mathematical model but it is inferred by observation:

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

This equation has the geometry of spacetime on the left side, and the content of the universe on the right side and shows how exactly matter curves spacetime.