The Hidden Language inside Music

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Introduction

I love music. I think everyone loves music. In fact, I really don't think I've ever met someone who truly dislikes music. Even if you don't have a favourite artist, I reckon you'd rather listen to something pleasant than nothing at all.

That last sentence is what got me thinking: what makes music pleasant? Why are some combinations of frequencies melodic, and others jarring? How on earth does an equation for heat dispersion come into this? If you choose to join me on this journey through music, maths, and a certain man named Joseph Fourier, then I assure you, all these questions (and more!) will be answered…

Setting the scene

I'm sitting in my piano lesson next to my teacher. She's talking, mainly to herself, about what it means for a piece to be played *with buoyancy*. It's in one ear and out the other.

"This A here, at the start of the bar, is supposed to be angular—"

"Miss Juliet, why does an A on a piano sound different to an A on a violin if they're both 440Hz?" I interrupt.

"What do you mean? They're the same pitch, they're both concert tuned."

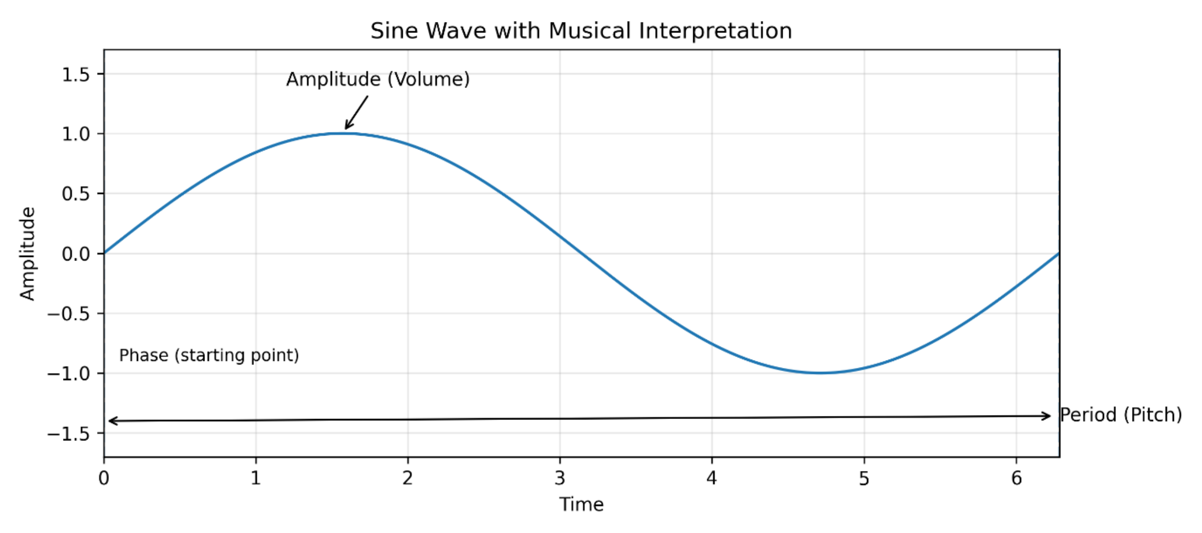
"I mean a violin sounds like a violin but a piano sounds like a piano."

"Ah… My dear, that is called *timbre*, and if you continue doing theory, we'll learn more about it. Now, look at these open fourths and fifths…"

Little did my piano teacher know, my being told of a funny French word—timbre—would be the catalyst of a research deep-dive into sound waves, Fourier analysis, and psychoacoustics.

What is music, really?

You're likely familiar with the notion of a sound *wave*. Musical terms—pitch, tone, volume—can be explained using the properties of waves, in this case frequency and amplitude.



Now, if you've ever seen a sound wave, they look nothing like the clean sound waves we're used to in mathematics—they look inextricable, messy, irregular; if they weren't periodic, you'd never think that they were waves in the first place. The field of harmonic analysis investigates the relationship between functions and their representations in terms of frequencies, typically involving Fourier series and transforms.

So, if music is just a function of air pressure over time, then perhaps we can analyse it. Not artistically, but *mathematically*. In fact, the Fourier series is exactly what we're going to use to analyse our sound waves.

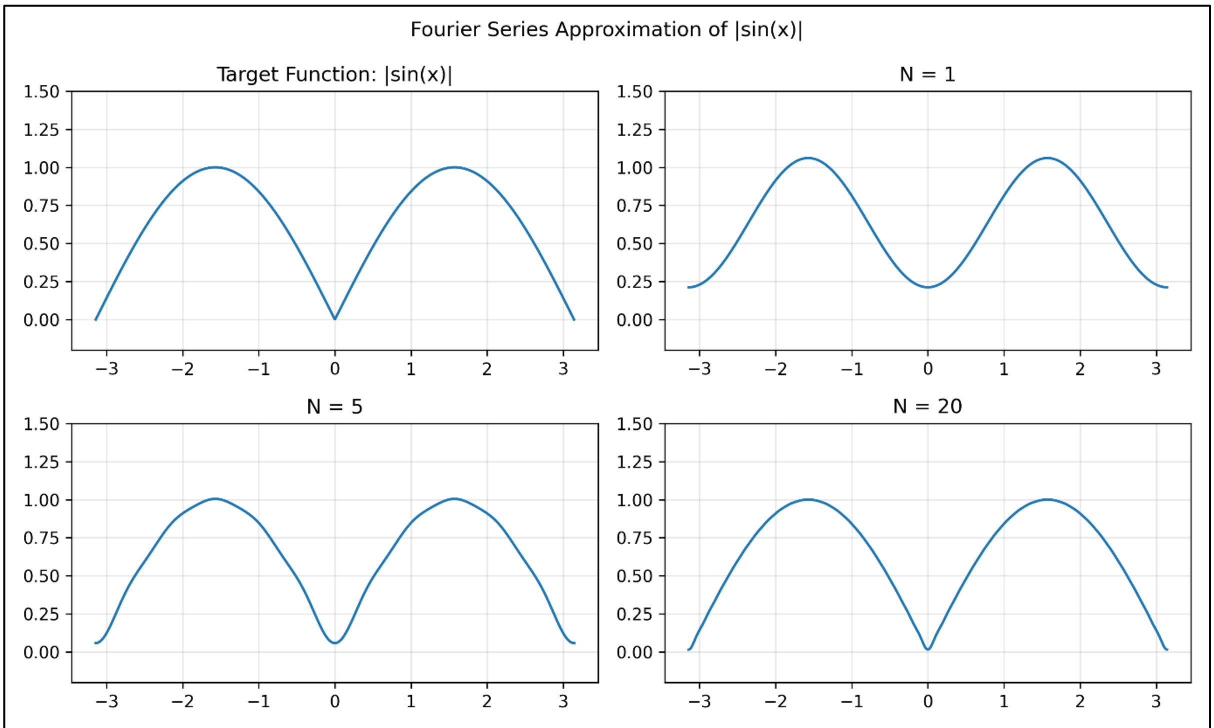


Figure 1: A Fourier series approximation of $|\sin x|$, *with the Fourier series being* $f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right)$

The Fourier series and derivation

Isn't that quite a bold claim, though? I mean, with all due respect, Mr Fourier, claiming that *all* periodic functions can be expressed as the sum of its Fourier series seems a bit assertive (just like calling it a *Fourier* series!). If you thought the same thing as me, reader, then it may be some consolation to know that we're not alone. Famous contemporary mathematicians like Pierre-Simon Laplace and Joseph-Louis Lagrange *(wonderful names)* viewed the claim as *nigh-heretical*.

Let's examine what Fourier said more closely: "Any reasonable function can be written as a sum of sine waves." This included jagged and discontinuous graphs—shocking, right? At the time, mathematicians expected functions to be smooth and well-behaved. After all, how can something with sharp corners be expressed using smooth, continuous sine waves? Are you sure that summing *to infinity* will give a real, working function? Also, Fourier derived his results from studying heat flow—where's the rigour?

Proofs and rederivations would come later, of course, from mathematicians like Johann Peter Gustav Lejeune Dirichlet *(another wonderful name)*. For now, we'll have a look at how Fourier got his result.

It all starts with heat, not music. Imagine a one-dimensional metal rod of length L with one end hot and one end cold (but fixed at the very ends so temperature = 0 there). We know that heat goes from where it is to where it isn't, spreading out and smoothing the temperature curve. Let's define a multivariable function $T(x, t)$ to describe temperature. If that adjective worries you, just remember that it means temperature depends on both where you are in space (hot/cold at either end) and time (heat disperses). The heat equation is:

$$\frac{\partial T}{\partial t} = k\frac{\partial^2 T}{\partial x^2}$$

The funny-looking "d"s might be new to you, but they just mean the same as a derivative and indicate that the function being differentiated depends on more than one variable. Essentially, this equation tells us that how fast temperature changes is proportional to how *"curved"* the temperature is.

Fourier was interested in solving this equation for $T(x, t)$. Unfortunately for him, this is a partial differential equation, and these are notoriously difficult to solve. Now, you might be familiar with a technique to solve *ordinary* differential equations: separation of variables. As it turns out, we can do the exact same thing here—we assume the form of the solution to be a product of two functions $f(x)g(t)$ and use this structure to solve the equation. Let us proceed!

Our boundary conditions are:

$$T(0, t) = T(L, t) = 0$$

And our assumed form is:

$$T(x, t) = f(t)g(x)$$

Substitute it back into the heat equation:

$$g(x)\frac{df}{dt} = kf(t)\frac{d^2g}{dx^2}$$

$$\frac{1}{f}\frac{df}{dt} = k\frac{1}{g}\frac{d^2g}{dx^2}$$

The LHS is a function of t, while the RHS is a function of x. Since they're equal, they must equal a constant. Let's call this constant $-\lambda$. With some rearranging, we obtain (for the spatial equation):

$$\frac{d^2g}{dx^2}+\frac{\lambda}{k}g=0$$

Look familiar? As it turns out, this is just the equation for a simple harmonic oscillator, which becomes more obvious if we let $\mu^2=\frac{\lambda}{k}$. Thus, the general solution is:

$$g(x)=A\cos(\mu x)+B\sin(\mu x)$$

Using the boundary conditions, $g(0)=0\Rightarrow A=0$ and $g(L)=0\Rightarrow \sin(\mu L)=0$. Thus,

$$\mu L=n\pi \quad (n=1,2,3,\dots)$$

and $g_n(x)=B_n\sin(\frac{n\pi x}{L})$. Hooray! Now let's solve the time equation:

$$\frac{df}{dt}=-\lambda f$$

This one is a lot easier to solve, and works out to be:

$$f_n(t)=e^{-\lambda t}=e^{-k\left(\frac{n\pi}{L}\right)^2t}$$

Combining our solutions gives us:

$$T_n(x,t)=\sin(\frac{n\pi x}{L})\,e^{-k\left(\frac{n\pi}{L}\right)^2t}$$

And, crucially, since derivatives are linear, we can add up all the solutions:

$$T(x,t)=\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{L})\,e^{-k\left(\frac{n\pi}{L}\right)^2t}$$

Looking at this, the initial temperature distribution ($t=0$) is just the Fourier sine series we saw earlier.

$$T(x,0)=\sum_{n=1}^{\infty}b_n\sin(\frac{n\pi x}{L})$$

Okay, that took a long time—so long, in fact, that you may have forgotten why we started this in the first place! It's now time to return to music and see why this is useful.

Using the Fourier series to analyse music

What's heat got to do with this? Well, the mathematics we used isn't restricted to temperature alone. After all, sound waves are just other functions, but this time they describe how air pressure varies over time.

If we *can* decompose such functions into sine waves of different frequencies, then maybe we can use this to disentangle the seemingly inextricable sound waves we encounter every waking moment. So, let's write down the Fourier series for a sound wave:

$$f(t)=\sum_{n=1}^{\infty}(a_n\cos(2\pi nf_0t)+b_n\sin(2\pi nf_0t))$$

Where f_0 is the fundamental frequency (e.g. 440Hz), nf_0 are the successive harmonics, and a_n is the amplitude of each harmonic. Now, we can start to unpack what *timbre* is! When two instruments play the same note, we're referring to the fundamental frequency f_0. However, the instruments have different coefficients a_n. The entire identity of the instrument is encoded in these; a sort of *musical fingerprint*. A flute will sound different to a violin, and both will be different to a piano.

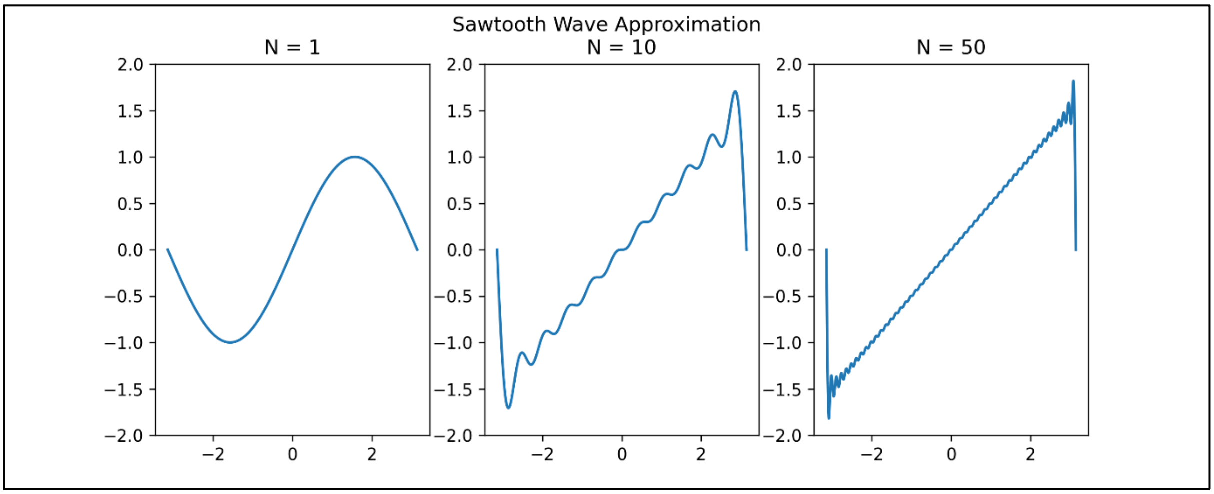
Coefficients aren't the only things that change, either—the waveforms themselves are also different. You've likely heard a sine wave before, which is just a singular frequency (like a beep) and does not change over time. It's quite rare for an instrument to produce these, but tuning forks, very pure flutes, and electronic oscillators are all capable. Their Fourier series is simply:

$$f(t) = \sin(2\pi ft)$$

As you can see, there are no harmonics and no timbre. Interestingly, distorted sine waves are used to make a very famous sound, the 808 bass. If you've listened to a pop, EDM, or hip-hop song, chances are that you've heard this sound before. Iconic!

Conversely, bowed string instruments (violins, cellos, etc.) produce sawtooth waves—rich, bright, and buzzy (no pun intended) sounds. Their Fourier series is:

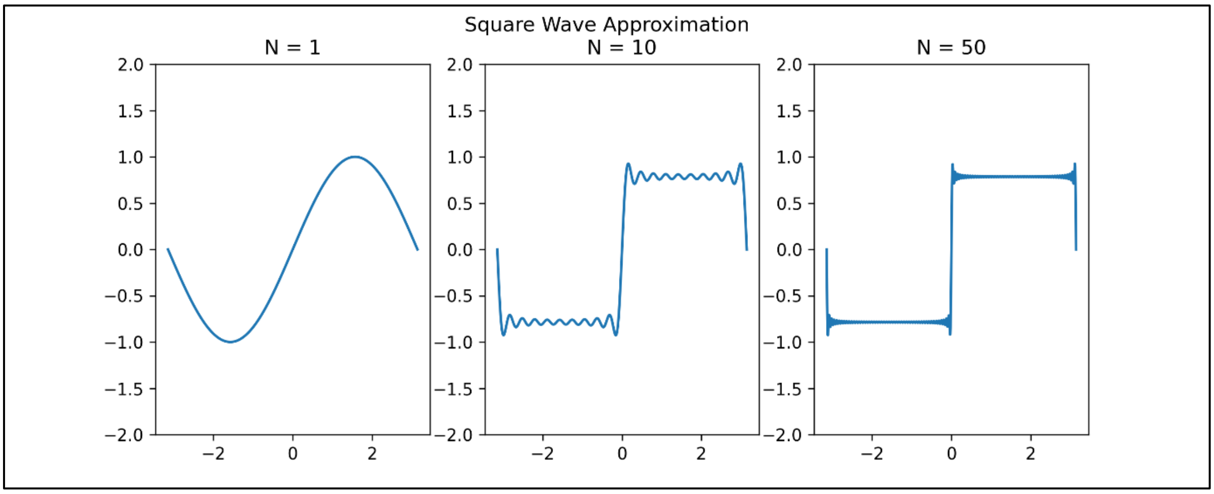
$$f(t) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin(2\pi nft)$$



This series contains all the harmonics, with odd ones added and even ones subtracted, and their amplitudes being the inverse of the harmonic. Use of a bow when playing these instruments is the reason why they have this waveform, contributing to their rich, full sound.

Square waves are another type of wave and can frequently be found in electronic music, particularly for 8-bit chip tune sounds, as well as the clarinet being an approximate instrument for these kinds of waves.

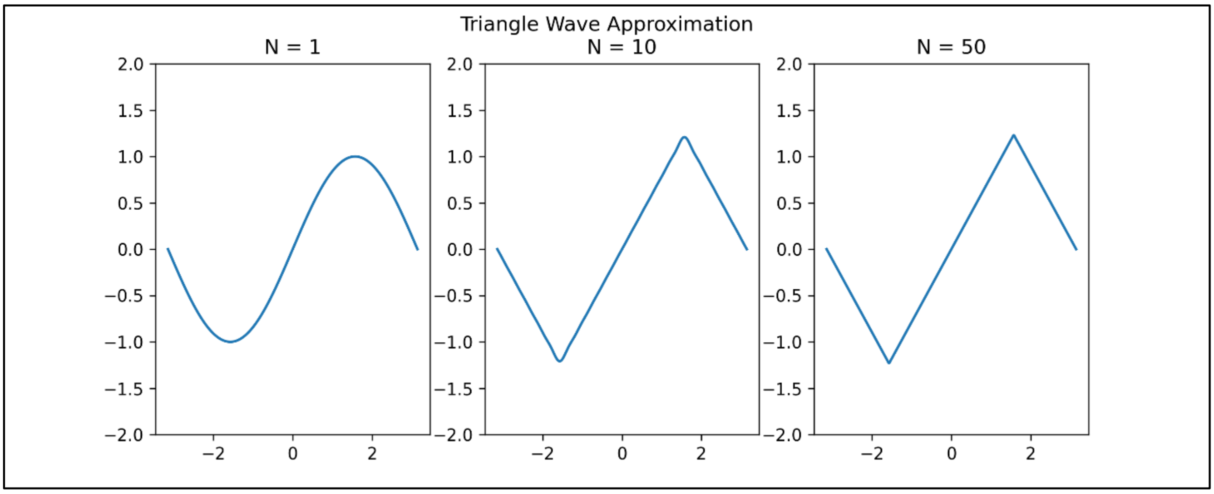
$$f(t) = \sum_{n=1}^{\infty} \frac{\sin(2\pi(2n-1)ft)}{2n-1}$$



Notice that square waves consist of only odd-numbered harmonics. The absence of even harmonics is what gives square waves their distinct "hollow" sound.

This leaves one last wave type, being the triangle wave. Notes from a flute are an approximate for these waves, particularly pure tones, and you may also hear them used in leads. Their Fourier series is:

$$f(t) = \sum_{n=1}^{\infty} (-1)^n \frac{\sin(2\pi(2n-1)ft)}{(2n-1)^2}$$



You can see that triangle waves only contain odd harmonics (similarly to square waves), but this time decaying much faster by $\frac{1}{n^2}$. Finally, we have our explanation for timbre—though instruments can play the same fundamental frequency, they have different *harmonic structure*. Violins have many strong harmonics, making a bright sound; clarinets have only odd harmonics, making a hollow sound; flutes have few weak harmonics, making a pure sound.

Consonance and dissonance

Now we understand why instruments sound different, but not why some combinations of notes sound pleasant, and others not. We call this *consonance* (pleasant) and *dissonance* (unpleasant). When we play two notes together, by superposition, they combine into a single waveform:

$$f(t) = \sum A_n \sin(2\pi n f_1 t) + \sum B_m \sin(2\pi n f_2 t)$$

And these notions of consonance and dissonance depend on the interactions of the harmonics: coincident/far apart frequencies cause consonance, while interference and beating cause dissonance. Remember that notes are not single frequencies, but whole *harmonic series*.

Perfect consonance occurs with an octave, where the two notes have respective frequencies of f and $2f$. Looking at the harmonics,

1. First note: $f, 2f, 3f, 4f, \ldots$
2. Second note: $2f, 4f, 6f, \ldots$

Almost every harmonic aligns, and we have a nice, smooth consonant sound with no beating. Another highly consonant sound is the perfect fifth, with frequencies of f and $\frac{3}{2}f$. These cause many low harmonics to coincide.

The tritone is arguably the most dissonant interval, where (in equal temperament) the ratio is *irrational*, being $1:\sqrt{2}$. That simple ratios cause consonance can be explained by the fact that we humans interpret frequency and pitch having an exponential relationship—the same perceived difference in pitch is not due to absolute increases in frequency (e.g. +200Hz), but rather the same ratio between successive frequencies (e.g. 2:3).

We know that the ratio between octaves is 1:2, and since there are 12 semitones in an octave, to go up one semitone we must multiply the frequency by $\sqrt[12]{2}$. This explains our consonant and dissonant frequencies:

1. Perfect fifth (7 semitones): $\left(\sqrt[12]{2}\right)^7 \approx \frac{3}{2}$
2. Perfect fourth (5 semitones): $\left(\sqrt[12]{2}\right)^5 \approx \frac{4}{3}$
3. Tritone (6 semitones): $\left(\sqrt[12]{2}\right)^6 = \sqrt{2} \approx 1.414$

Conclusion

Though the digital world may be boundless, real-world limitations force all of us, musician or not, to never sing a single frequency, but a whole orchestra (pun intended) of frequencies. We now know why instruments sound different, why music sounds good, and just how smart that Fourier guy was.

Decomposing a function into a sum of sine waves may suggest that perhaps waves are fundamental not only in mathematics and music, but anything involving functions. In fact, as it turns out, everything with momentum has a wavelength (according to de Broglie). So, the maths that describes a violin note also describes the universe around us.

Does understanding music mathematically make it less beautiful, or more? On one hand, we have demystified why music is nice and what sound actually is. Nevertheless, we still have many unanswered questions. Personally, I think it's a wonderful scientific explanation behind music, but the *ultimate* reason why we like music is determined by *us*, our perception—something distinctly human and flavourful.

Reader, thank you for reading my essay. I hope you now appreciate music better and feel inspired to be a musician of your own!