The mathematics of a sponge cake.

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Introduction

I have always been a big fan of baking, and maths, so when thinking of a topic for this essay, this was a clear option. I am going to take you through some of the process I use to make a sponge cake whilst overanalysing each step of the process - including measuring ingredients, selecting a tin, baking and serving the cake - and looking at them mathematically.

Preparing and weighing ingredients

We are going to need many ingredients, such as flour, sugar, eggs, etc. But, the ingredient I want to focus on is baking powder, specifically how it is measured. The recipe I use calls for $\frac{1}{2}$ tsp of baking powder, but I only have a full teaspoon. Now most people would just eyeball it and fill it about halfway, but I want my cake to be mathematically perfect.

I'm going to assume that a teaspoon can be modelled as a hemisphere, but how do we find its volume? Here we can use volumes of revolution. If we take a function $f(x)$, we can find the area under it and rotate it around the x axis. The formula for this volume from a to b is:

$$\pi \int_a^b y^2 \, \mathrm{d}x.$$

To find the volume of a hemisphere this function would be a quarter circle, with the equation $x^2 + y^2 = r^2$

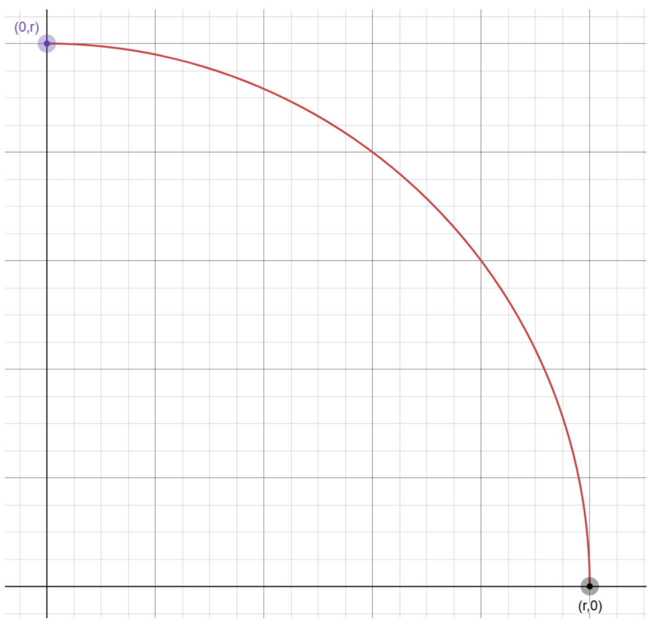

Figure 1: The quarter circle

Using the formula and the rearranged equation, $y^2 = r^2 - x^2$, we evaluate the volume of a hemisphere, using the bounds 0 and r.

$$\pi \int_0^r r^2 - x^2 \, \mathrm{d}x.$$

r^2 is a constant, so its antiderivative is r^2x and $-x^2$ integrates to $-\frac{1}{3}x^3$.

$$\pi \int_0^r r^2 - x^2 \, \mathrm{d}x = \pi \left[r^2x - \frac{1}{3}x^3 \right]_0^r$$

So, we find an expression for the volume:

$$V = \pi \left[r^2x - \frac{1}{3}x^3 \right]_0^r = \pi \left(r^3 - \frac{1}{3}r^3 \right) = \frac{2}{3}\pi r^3$$

We now know the volume of the hemisphere, so our desired volume is, $\frac{1}{3}\pi r^3$, which is found from the bounds 0 to a, so, to find the proportion the spoon is to be filled up to we find $\frac{a}{r}$.

$$\pi \left[r^2x - \frac{1}{3}x^3 \right]_0^a = \pi \left(r^2a - \frac{1}{3}a^3 \right) = \frac{1}{3}\pi r^3$$

By dividing by π and rearranging, we can create the expression.

$$a^3 - 3ar^2 + r^3 = 0$$

We can then divide through by r^3 to find an expression in terms of $\frac{a}{r}$

$$\frac{a^3}{r^3} - 3\frac{a}{r} + 1 = 0$$

If we let $k = \frac{a}{r}$, we create a cubic in terms of k, $k^3 - 3k + 1 = 0$, and now we can find its roots. Here we can use a technique, Vieta's substitution, where we let $k = z + \frac{1}{z}$. If we substitute, and expand, we get the expression:

$$z^3 + 3z + \frac{3}{z} + \frac{1}{z^3} - 3z - \frac{3}{z} + 1 = 0$$

The z and $\frac{1}{z}$ terms cancel,

$$z^3 + \frac{1}{z^3} + 1 = 0.$$

So, now we form a quadratic in terms of z^3 by multiplying by z^3.

$$z^6 + z^3 + 1 = 0$$

If we use the quadratic formula, we can determine the roots of the quadratic, where we obtain, $\frac{-1 \pm \sqrt{3}i}{2}$. These are the roots of unity of $z^3 = 1$, so the roots are also $e^{\frac{2\pi}{3}i}$ and $e^{\frac{4\pi}{3}i}$. We know these are equal to z^3, so, taking cube roots, we find the values of z to be

$$z = e^{\frac{2\pi}{9}i}, e^{\frac{8\pi}{9}i}, e^{\frac{14\pi}{9}i},$$
$$e^{\frac{4\pi}{9}i}, e^{\frac{10\pi}{9}i}, e^{\frac{16\pi}{9}i}$$

Now finally, to find k, we know $k = z + \frac{1}{z}$, and z is in the form $e^{i\theta}$, so $k = e^{i\theta} + e^{-i\theta}$ and, using Euler's formula.

$$e^{i\theta} = \cos(\theta) + i\sin(\theta)$$

We can find that,

$$k = 2\cos(\theta)$$

then by substituting our values of θ, we can find our values of k to be:

$$k = 2\cos\left(\frac{2\pi}{9}\right), 2\cos\left(\frac{4\pi}{9}\right), 2\cos\left(\frac{8\pi}{9}\right).$$

However, only $2\cos\left(\frac{4\pi}{9}\right)$ gives a value < 1, so our value of $\frac{a}{r}$ is approximately 0.34729..., but, this value is from 0 to a and so gives us the proportion from the top of the spoon. So our real answer is $1 - 0.34729...$ which means the spoon must be filled to around $\frac{2}{3}$ (0.6527...) of its height to provide $\frac{1}{2}$ of its volume, the amount of baking powder we need for our cake.

The tin

The obvious next step is to mix my ingredients together, but first I need to select the correct tin. I know that we will create a fixed volume of cake mix, but, I want to minimise the surface areas of the cake. The tin can be modelled as a cylinder, so, its volume is given by $V = \pi r^2 h$, and the top and side surfaces are given by $A = \pi r^2 + 2\pi r h$. As volume is fixed, I want to express A in terms of V and h. So let's rearrange the volume formula to find:

$$\frac{V}{h} = \pi r^2$$

and,

$$r = \sqrt{\frac{V}{\pi h}}.$$

We can then substitute these into our formula for surface area,

$$A = \frac{V}{h} + 2\pi h\sqrt{\frac{V}{\pi h}}$$

and simplify

$$A = \frac{V}{h} + \sqrt{\frac{4\pi^2 h^2 V}{\pi h}} = \frac{V}{h} + \sqrt{4\pi h V}$$

Now, as we want to minimise the surface area we can differentiate this expression and find the minimum value by equating to 0.

$$\frac{\mathrm{d}A}{\mathrm{d}h} = -\frac{V}{h^2} + \frac{2\pi V}{\sqrt{4\pi h V}} = 0$$

We can rearrange, and multiply by h^2,

$$V = \frac{2\pi V h^2}{\sqrt{4\pi h V}}$$

Now, we square each side and cancel terms,

$$V^2 = \frac{4\pi^2 V^2 h^4}{4\pi h V} = V\pi h^3$$

So, we find that

$$V = \pi h^3$$

And therefore,

$$\pi h^3 = \pi r^2 h$$

So, for the minimum surface area required, $r = h$, so we'll use a tin where the radius is equal to the height.

Baking the cake

Now we've mixed all our ingredients, the next step is to bake the thing! Obviously, I'm going to look mathematically at this, so I'm going to model the volume of the cake as a function of time. As the cake is put in the oven, its initial volume is that of the tin: $V_0 = \pi r^2 h$, but, in the last section, we found $r = h$ for our tin, so $V_0 = \pi r^3$. As the cake bakes, the top will expand into a domed shape, adding additional volume. I am going to model this dome using a paraboloid, with equation in the 3D axis as:

$$z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}\right)$$

Here c will change with time, stretching the paraboloid upward whereas a and b stretch in the x and y direction respectively and, as we want the base to be circular, we let $b = a$:

$$z = c\left(1 - \frac{x^2}{a^2} - \frac{y^2}{a^2}\right)$$

The volume of a paraboloid like this can be defined with the equation:

$$V_p = \frac{\pi}{2} a^2 c$$

So our overall volume at time T is defined as:

$$V_T = V_0 + V_p = \pi r^3 + \frac{\pi}{2} a^2 c$$

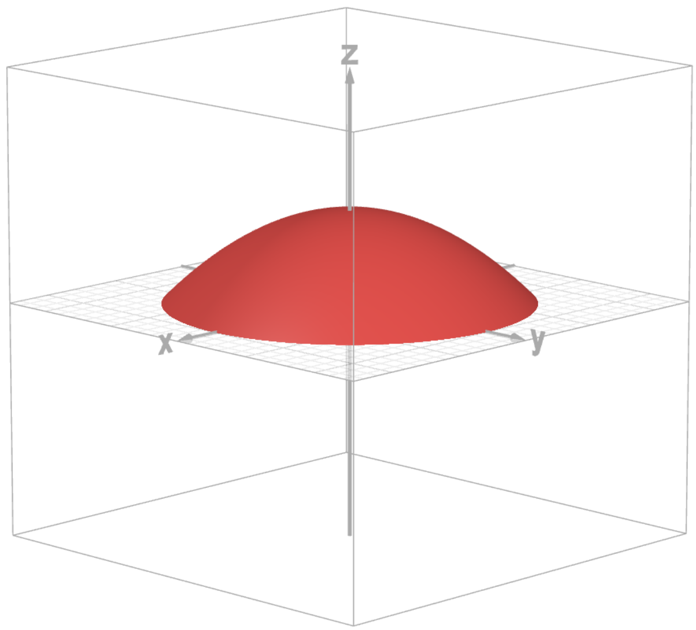

Figure 2: The paraboloid at c = 0.5

Now we want to get this equation entirely in terms of r and t (time). Firstly, we know that a, as the stretch, is therefore equal to the radius, r, so all we need is to find c. Here, I am going to make an assumption, one that is probably completely wrong, but fun mathematically for me, that c is proportional to the temperature of the cake, θ, so:

$$c = k\theta$$

So, our next goal is to find an expression for θ in terms of time. Here I will set up a differential equation, where the rate of the increase of the cake's temperature is proportional to the temperature of the oven (θ_α) minus the temperature of the cake (θ).

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = \lambda(\theta_\alpha - \theta)$$

We can solve this using separation of variables:

$$\frac{1}{(\theta_\alpha - \theta)}\mathrm{d}\theta = \lambda \mathrm{d}t$$

And we integrate both sides to find,

$$-\ln(\theta_\alpha - \theta) = \lambda t + C$$

Then we rearrange for θ to find that:

$$\theta_\alpha - \theta = Ae^{-\lambda t}$$

$$\theta = \theta_\alpha - Ae^{-\lambda t}$$

For baking a sponge cake, θ_O is around 180°C and if we let $t = 0$ we can find that for an initial temperature of 20°C, A is 160, so:

$$\theta = 180 - 160e^{-\lambda t}$$

Therefore,

$$c = k\left(180 - 160e^{-\lambda t}\right)$$

and so at time t (minutes) the volume of our cake is given by:

$$V_T = \pi r^3 + \frac{\pi}{2} r^2 k\left(180 - 160e^{-\lambda t}\right)$$

Slicing the cake

Finally, now our cake is fully baked and ready to be eaten, the final stage is slicing it into pieces. In this section, I want to explore how to cut a cake into a certain number of equal pieces, and to try and do it in the minimum number of slices. To simplify the problem, I will model the cake as a cylinder.

The first simple theorem we can create is as follows: *A cake can easily be cut into n pieces with n slices.* This is easy to show, as it can be done be slicing the top face of the cylinder into sectors with an angle of $\frac{2\pi}{n}$, for example for 4 pieces, 4 slices to the centre of the cake at angles of $\frac{\pi}{2}$ to each other provides 4 pieces.

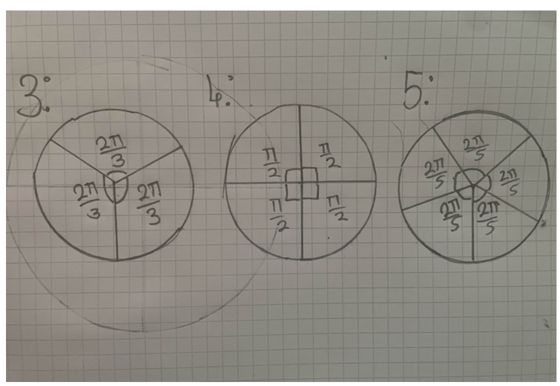

Figure 3: Examples of theorem 1 for 3, 4 and 5 pieces.

However, this is clearly not the best method, as 4 pieces can obviously be done in only 2 slices. This brings me to my second theorem: *To cut a cake into an even number of pieces, ($2n$), this can be done in n slices.* This is a simple continuation of the last theorem as to do this all that must be done is for the slices to be extended to the opposite edge of the circular top for any odd number,

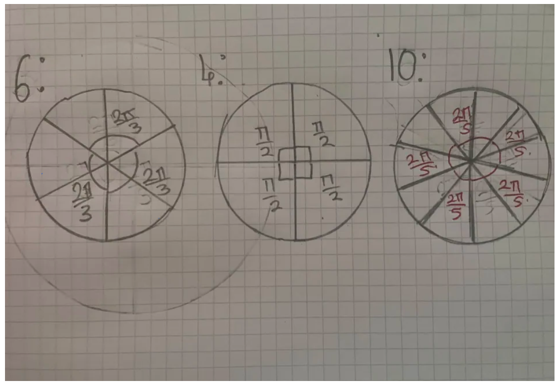

Figure 4: Circles in figure 3 repeated, with radii of 3 and 5 pieces extended to create 6 and 10 pieces.

thus creating diameters spaced $\frac{2\pi}{n}$ apart. This idea works for all none powers of 2, but they can be done easily by just halving the cake over and over.

Now, we want to be able to extend this for odd numbers. To do this we will have to come up with a new proof for theorem 2. For this proof, instead of cutting across the circular top, we will cut along the curved edge of the cylinder. If we take the height to be 1, for an even number q, this can be split into a group of fractions,

$$\frac{2}{q} + \frac{2}{q} + \frac{2}{q} + ...$$

with $\frac{q}{2}$ pieces. For example, for 10 pieces, the curved edge is cut into 5 pieces each with a height of $\frac{2}{10}$. Then, these pieces can be stacked and then cut in half all together. For the first set of cuts there will be $\frac{q}{2} - 1$ slices, as it takes 1 less than the number of pieces. Then there will be 1 extra slice for a total of $\frac{q}{2}$ slices for an even number q.

To extend this for an odd number q, we instead split the height to a slightly different group of fractions,

$$\left(\frac{2}{q} + \frac{2}{q} + \frac{2}{q} + ...\right) + \frac{1}{q}$$

There will be $\frac{q-1}{2}$, $\frac{q}{2}$ terms and 1, $\frac{1}{q}$ terms. So the number of slices for these fractions is $\frac{q-1}{2}$. Once again we stack the pieces with height $\frac{2}{q}$ and halve them. This leads to a total number of slices to be $\frac{q-1}{2} + 1$, which simplifies to a total of $\frac{q+1}{2}$ slices.

So we can conclude that: *For any odd number of pieces, q it will take the same number of slices as the next even number of pieces ($\frac{q+1}{2}$).* For example, we can cut a cake into 17 pieces with only 9 slices.

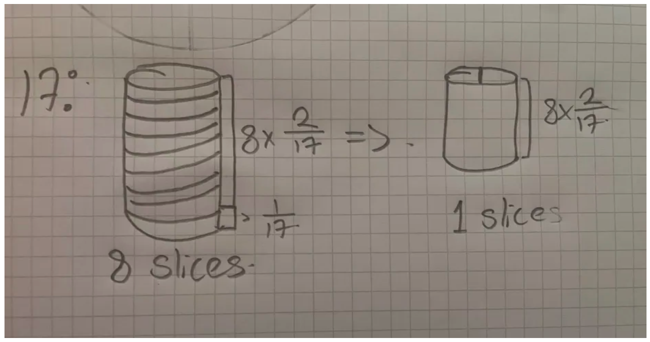

Figure 5: Method to cut a cake into 17 pieces with 9 slices.

Okay, so far we have come up with a reasonably effective method to minimise slices, but we can do better.

Using the same logic we discovered for the last theorem for odds, we can further decrease the number of slices by using a different even number for the numerator of our fractions. Here I created a hypothesis: *If an odd number n, mod $2k$ equals 1 then then splitting n into pieces of the form $\frac{2k}{n}$ and $\frac{1}{n}$ is the best way to minimise slices.* Our previous theorem is just a case of this for $k = 1$ We can see that it works for different value of k with our last example, we could have split the height into 4 pieces of length $\frac{4}{17}$ and 1 piece of length $\frac{1}{17}$, which would take 4 slices, then 2 more, to divide our $\frac{4}{17}$ pieces up, for a total of only 6 slices.

So lets find a general formula for the number of slices. Essentially, our odd is in the form $2p + 1$, and so if our even number $2k$, has mod of 1, then $k \mid p$. We know the height is split into a group of fractions:

$$\left(\frac{2k}{2p+1} + \frac{2k}{2p+1} + \frac{2k}{2p+1} + ...\right) + \frac{1}{2p+1}$$

Using our example, we can see the number of pieces in the form $\frac{2k}{2p+1}$ is $\frac{p}{k}$ and so it will take that many slices for the initial split. Then we can stack all our pieces bar $\frac{1}{2p+1}$ and, as they need to be split an even amount (each part into $2k$ pieces), we use our previous formula to see that it will take k slices. This means the total number of slices is:

$$S_k = k + \frac{p}{k}$$

We know there are likely many values of k which divide p, so we want to minimise $k + \frac{p}{k}$. We do this by differentiating with respect to k, and setting equal to 0:

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(k + \frac{p}{k}\right) = 1 - \frac{p}{k^2}$$

$$1 - \frac{p}{k^2} = 0$$
$$k^2 = p$$

So it is minimised when,

$$k = \sqrt{p}$$

Using all this we can create a new theorem: *To find the minimum number of slices to create* $2p+1$ *pieces, find the factor of* p *(*k*) closest to* $\sqrt{p}$ *and substitute into the formula,* $S_k = k + \frac{p}{k}$. For example, according to our formula for 65 pieces, the factor of p (32) closest to $\sqrt{p} = \sqrt{32} \approx 5.66$ is 4, so we should be able to cut into 65 pieces with only

$$4 + \frac{32}{4} = 12$$

slices.

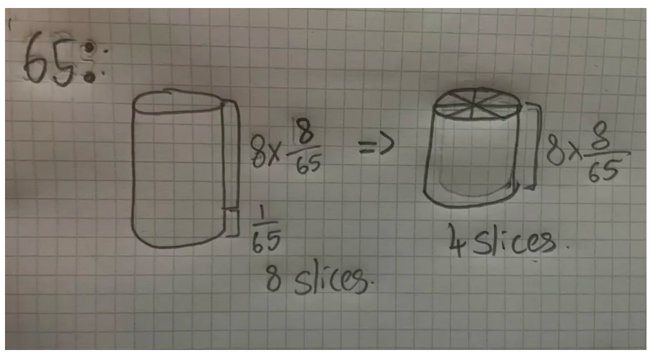

Figure 6: Method to cut a cake into 65 pieces with 12 slices.

We are almost done now, we just need to extend this to even numbers. For evens, we will once more split 1 into a group of fractions. For an even number $2p$ and a factor k of $2p$, we will split the height into:

$$\frac{k}{2p} + \frac{k}{2p} + \frac{k}{2p} + ...$$

So, there will be $\frac{2p}{k}$ pieces, so it takes $\frac{2p}{k} - 1$ slices. Then we will need to split these into $\frac{1}{2p}$ while stacked, so we can only slice across the top face. This means when k is even it will take $\frac{k}{2}$ slices, from our 2nd theorem, and if odd it will take k slices, from our 1st theorem. So we always want k to be even. Therefore the total number of slices will be:

$$S_k = \frac{k}{2} + \frac{2p}{k} - 1$$

We want to minimise this, so again we differentiate and set equal to 0:

$$\frac{\mathrm{d}}{\mathrm{d}k}\left(\frac{1}{2}k + \frac{2p}{k} - 1\right) = \frac{1}{2} - \frac{2p}{k^2}$$

$$\frac{1}{2} - \frac{2p}{k^2} = 0$$
$$k^2 = 4p$$

So, slices minimised when,

$$k = 2\sqrt{p}$$

Finally, we create our last theorem: *The minimum slices needed to create $2p$ pieces, find the factor of $2p$ closest to $2\sqrt{p}$ and substitute into the formula,* $S_k = \frac{1}{2}k + \frac{2p}{k} - 1$.

What have we learned?

Well, creating this essay has helped me to learn some new maths and apply it to one of my hobbies in a 'practical' way. In the end I don't really think I will be using this the next time I make a cake, but at least I had fun and you (hopefully) learned something reading this.