The Mathematics of Oil Paints

Picture this: you're perusing the grand halls of The Barber Institute of Fine Arts in Birmingham (any other gallery would work just fine, but in this scenario, you live around here and don't have the budget to peruse any further), and you admire some of the most incredible artworks ever to have graced the canvas. You watch generations pass with a single step; in seconds, you've experienced life across seven centuries, through the eyes of Bellini, Botticelli, Manet - all the greats. You've never considered yourself to be the kind of person who enjoys strolling around museums as a recreational activity, but hey, people change, and you're excited to unleash the creativity that you know has always lived within you, and - oh.

Suddenly, your inner analyst has awakened, and you can't help but notice an almost eerie sense of familiarity that has followed you since you laid eyes on the first painting. Perhaps it's the distinct smell of the gallery - like dust and linseed oil. Or the dim, pasty light blaring from the bulbs overhead. Or maybe, just maybe, you're being haunted by the ghost of Van Gogh himself…

Nope. That's not it.

It takes you hours of pacing the long rooms, reading the art descriptions, and inhaling the musty smells of the paints before you finally notice that almost every piece carries the same smooth quality, the same texture. Can you guess what I'm hinting towards?

It has been a long-held belief that oil paints - perhaps the most common medium used in the fine arts (and also the reason art galleries give you intense deja vu) - are the easiest paints to use to achieve realism in paintings. I hope I've managed to hook you with this introduction, because today I plan on formalising this unspoken law among artists using mathematical modelling (and a bit of creativity).

1 - Introduction

Painting realistic pieces can be seen as the process of reducing the error between a painting and the subject. Each brushstroke aims to minimise discrepancies in colour, shape and visual texture. If the aim of painting is to make the canvas appear as close as possible to the reference, then the act of painting becomes an iterative optimisation procedure:

$$\theta_{k+1} = \theta_k + \Delta\theta_k$$

Where is chosen to reduce error. $\Delta\theta_k$ is chosen to reduce error. Let's develop this framework

2 - Our model

Let $\Omega \subset \mathbb{R}^2$ denote the canvas domain. A reference image is a function:

$$T: \Omega \rightarrow \mathbb{R}^3$$

A painting is:

$$P_\theta : \Omega \to \mathbb{R}^3$$

Where $\theta \in \Theta$ is a finite dimensional parameter vector of the artist's possible choices. The mapping $\theta \mapsto P_\theta$ is non linear and dependent on the medium.

3 - Error

An artist’s goal is essentially to reduce perpetual discrepancies between a reference and their painting. (unless they paint abstract - I have no idea what’s going on there, but I guess that’s the point)

We can define the total error score $E(\theta)$:

$$E(\theta) = \int_\Omega \Phi(P_\theta(x), T(x))\, dx$$

If $E(\theta)$ is large, the difference between the reference and painting is also large. We integrate to check every single point of the painting against the reference and sum up all the discrepancies, where Φ measures the difference between the painting and the reference. If the colour values of a point on the painting differs from the colour value of the same point on the reference, Φ calculates this difference in 3D colour space.

For a realistic error function, we want to include several components:

$$E(\theta) = + \alpha E_{value} + \beta E_{chromatic} + \gamma E_{edge} + \delta E_{structure}$$

These components, value (lightness and darkness), chromatic (specific hues of colour), edge (the boundaries between depicted objects), and structure (overall shape), are examples of things an artist may take into account throughout the painting process. The artist can choose what components are a higher priority and when (I’m no artist myself, but I can safely assume getting the general shape of a subject is just a smidge more crucial than nailing the exact shade of terracotta hickory-oak brown.)

We end up with a multiscale structural term:

$$E_{structure} = \Sigma wi \int_\Omega // G_{\sigma i} * P_\theta (x) - G_{\sigma i} * T(x) //^2 dx$$

$$P_\theta : \Omega \to \mathbb{R}^3$$

Where $G_{\sigma i}$ is gaussian blur. 'What on earth does that mean?', you ask. This whole expression simply represents how artists usually start with blurry shapes before adding fine details.
$G_{\sigma i}$ is a function that takes a single speck of paint on the canvas (or a pixel) and averages it with the other surrounding pixels. We compare P_θ (x) and T(x) once $G_{\sigma i}$ has been applied. This means that instead of comparing the fine details of our painting and the reference image, we first compare the larger shapes in a process of abstraction. Squaring our result amplifies larger mistakes more so than smaller mistakes, modelling how an artist would prioritize correcting major issues in the painting before obsessing over minute details (there's enough time for that later; we will soon talk about why oil paints enable artists to be fussy with the small details - hint, it's to do with drying times).
We integrate to represent continuous addition of the squared errors across the canvas, and we sum them up to give us a value for structural error at a specific blur. Finally, the summation term Σwi calculates the total error for different scales of blurriness, the weight (wi) telling us what scale to care about the most. As an artist progresses with a painting, the size of the blur decreases (the artist focuses on big shapes first, then smaller shapes).

4 - constraints

If Bellini and Manet set out to produce an artpiece of exactly the same reference, it's safe to assume they wouldn't look the same. This is because of material constraints and differing personal styles, which you could call stylistic constraints.

We define:

$$R(\theta) = \lambda_1 R_{smooth}(\theta) + \lambda_2 R_{simplicity}(\theta) + \lambda_3 R_{style}(\theta)$$

R stands for Regularisation, and represents the limits of what can be done with the paint, and also the style of the artist. R(θ) increases if the artist breaks their own rules. For example if the painting isn't smooth enough (limiting the realism factor), R(θ) becomes larger. If λ_2 is large, the artist is placing large importance on being minimalistic.

The full optimisation problem then becomes:

$$\min F(\theta) = E(\theta) + R(\theta)$$

5 - gradient descent

During an iteration, k, the artist applies corrections:

$$\theta_{k+1} = \theta_k - \eta_k \nabla F(\theta_k)$$

∇F Represents a gradient. This is a vector which points in the direction in which error is increasing the most. For example if the artist senses a colour is off, like the wrong tone of blue, the gradient points towards 'blueness'. The gradient is subtracted. The artist identifies the direction of the error (the gradient) and makes a move to rectify it, (adding white paint to lighten the blue colour, for instance). This is iterated, reducing the overall error and making a more realistic painting. Essentially, painting can be seen as human guided gradient descent!

6 - Oil paint

Although it can be said oil paint is the most common medium because of conventions in the art world, which may be true, it could also be said that oil paint's properties made it the natural best choice; mathematically speaking oil paint has qualities that make it perfect for error minimisation. In our framework the medium used determines the geometry of the error landscape, the stability of parameter updates, and also the dimensionality of parameter space.

6.1

Oil paint dries by oxidation (this is a fancy way of saying very, very slowly.) This works to not only test your patience, but also increases the number of optimisation steps that can be made before a layer dries.

$$\theta_{k+1} = \theta_k - \eta_k \nabla F(\theta_k)$$

Revisiting this equation from earlier, η_k represents step size. Using oil paints, η_k is smaller and more stable. Other mediums with faster drying times force larger, clumsier jumps. The artist can slowly add more white paint to the blue, reducing the chance of adding too much and making an irreversible mistake.

6.2

All this talking about watching paint dry is - surprise surprise - getting boring. Let's switch things up a bit.

Oil paint enables the artist to utilise the 'wet on wet' method of painting (applying wet paint over an already wet layer - I know, the name doesn't leave much to infer). We can approximate this technique with the diffusion equation:

$$\frac{\partial P}{\partial t} = D\nabla^2 P$$

The right hand term represents the painting changing over time as the paint diffuses across the canvas. $\nabla^2 P$

Is the Laplacian. This term essentially takes a point on the canvas (like a pixel in a photo) and compares it to surrounding points, or pixels. A high Laplacian represents a stark difference between neighboring points. If an edge is too harsh (which will be seen as much darker or lighter points in comparison to surrounding areas) the $\nabla^2 P$ term will be large. Multiplied by a constant D (representing how runny or liquidy the medium is), the right hand term itself becomes large, showing a large change in the painting over time.
This explains how the wet-on-wet technique reduces noise in the painting leading to a more smooth and realistic outcome.

6.3

It would be rather strange to be working on a maths problem to find that your numbers seem to change unexpectedly, with no further input from you. It'd be likely that your calculations would go horribly wrong, even though you thought you'd nailed the job.
If you were an artist who stepped away from the canvas to allow your masterpiece to dry, and came back to find the paint has moved (be it due to the laws of physics, or the undying will of the universe to watch you suffer) you might just give up then and there. But fear not! Oil paints are coming to the rescue!

The formula:

$$\theta_k \mapsto \theta_k + \varepsilon$$

Represents the parameter drift as paints dry and move on the canvas. ε represents a small amount of error introduced. Watercolour, for example, has a very large ε due to its thinner consistency and the fact it dries by evaporation, resulting in colors changing shades as the water leaves the pigment over time. In comparison, oil paints have negligible ε, so:

$$\nabla F(\theta_k + \varepsilon) \approx \nabla F(\theta_k)$$

This simply means the artist will not fight for their sanity every time they finish a painting; the shapes and colours will remain just as the artist left them.

6.4

Oil paints has several ways that it can be used that vary the very dimensions of the medium. You can say that oil paints have the largest feasible parameter space of all mediums.

Oil supports:

- Variable opacity
- Variable gloss
- Variable thickness
- Unlimited blending time

And more. The formula:

$$\dim(\Theta \text{ oil}) > \dim(\Theta \text{ acrylic}) > \dim(\Theta \text{ watercolour})$$

summarises this. Θ represents parameter space. A higher dimensional space enables artists to find better solutions to problems. Oil paint simply has more ways it can be used and manipulated to reach a desired effect.

Conclusion

In the process of writing this essay, I can confidently say I've acquired a wider appreciation for art, and, (believe it or not), have developed an affinity for all mediums - I couldn't just conclude only singing praise for oil paints. Although possessing many strengths, oil paints are notoriously expensive, perhaps in part due to the properties we've discussed. This explains why the growing popularity of other mediums like acrylic, watercolour, gouache, pastels (the list is endless…), has also seen a widened participation in creative endeavors amongst people from all socio-economic backgrounds. The best artists are no longer just the ones who had access to expensive classes and materials, but those with passion and a drive to improve no matter the difficulties faced (including difficult mediums - watercolour, I'm looking at you.)

Although from a purely objective and even mathematical standpoint, some mediums are fundamentally more limited than others, there will always be one variable we cannot account for when discussing materials on a general basis - the skill of the artist.