Introduction

I love to play sports, but some of them cause my appetite to drop due to the adrenaline rush. For instance, I once attended 3 hours of Brazilian Jiu-Jitsu classes in the evening after skipping breakfast and eating a roughly 600 calorie lunch. Estimates online state that training BJJ burns roughly 500-1000 calories per hour; I will treat that as 600 calories per hour as I am fairly light at 56kg. I normally eat 2100 calories on non-training days, so after eating a light dinner of about 400 calories, my caloric deficit for the day was roughly $2100 + (600 \times 3) - (600 + 400) = 2900$ calories.

That night, aside from suppressing my hunger, the adrenaline also prevented me from falling asleep, which led me to think about the mathematics of weight loss and caloric deficits.

Background

One pound of body fat contains approximately 3500 calories of energy. The first law of thermodynamics states that energy can neither be created nor destroyed, only transformed. The term "burning calories" specifically refers to potential energy stored in fat being converted into other forms, such as thermal and kinetic energy, thus getting rid of one's fat. Weight loss should be simple then – "burn" the same number of calories whilst "taking in" less calories.

A sedentary 20 year-old man who is 200cm tall and weighs 180kg (almost 400lbs) burns 3546 calories a day according to the Mifflin-St Jeor equation. Suppose this hypothetical man went fasted for 200 days, only drinking water and consuming minerals and vitamins (which are calorie-free); would this man lose 200lbs? If not, how much weight would he lose? Due to the danger and unethicality of performing such an experiment, we shall attempt to answer this question using mathematics.

Modelling Weight Loss

There are various methods to estimate one's Basal Metabolic Rate (BMR), the minimum number of calories your body needs to perform basic life-sustaining functions like breathing and circulation. The Mifflin-St Jeor equation is perhaps the most common formula for BMR and is shown below; define W as weight in kilograms, H as height in centimeters, and Y as age in years.

$$BMR\ (Men) = 10W + 6.25H - 5Y + 5$$

$$BMR\ (Women) = 10W + 6.25H - 5Y - 161$$

Figure 1: Mifflin-St Jeor Equation

Total Daily Energy Expenditure (TDEE) is a multiple of BMR and includes calories burnt through actions like exercising, walking and even fidgeting on top of BMR. For a sedentary person, a multiplier of $TDEE = 1.2 \times BMR$ is typically used.

Without performing explicit calculations, we can simply answer "no" to our first question. Across 200 days, a 200cm tall 20 year-old man is unlikely to grow much in height, and he will merely age by just over half a year. Hence, the change in BMR (and TDEE, which is directly proportional) due to changes in height and age are minimal. However, weight is linearly related to TDEE and since the man is expected to lose weight each day, his TDEE will also decrease over time.

Just as a physicist might approximate a penguin's energy radiation by modelling it as a cylinder,[1] we shall make some reasonable assumptions to simplify our analysis.

Define t as time in days and $E(t)$ as energy intake. Recall W and TDEE have already been defined. Weight loss entails a calorie deficit, hence we shall model the rate of change of weight over time to be proportional to the difference between energy intake and energy expenditure. This can be shown as a differential equation where k is a proportionality constant:

$$\frac{dW}{dt} = \ - k(E(t) - TDEE(W, t))$$

For someone who consumes zero calories, $E(t) = 0$, hence we can simplify the equation:

$$\frac{dW}{dt} = \ - k(TDEE(W, t))$$

According to a model utilized in a study done by the University of Cambridge, TDEE is equal to the sum of BMR,[2] Physical Activity Energy Expenditure (PAEE) and Diet-induced Thermogenesis (DIT).

$$TDEE(W, t) = BMR + PAEE + DIT$$

In our very hypothetical scenario, we can assume $DIT = 0$ as there is no food for the body to process. Although it is estimated that $TDEE = 1.2 \times BMR$ for a sedentary person, the

[1] This is in reference to an old meme of a physics textbook stating, "assume that a penguin is a circular cylinder."

[2] The study used the term Resting Energy Expenditure (REE) instead of BMR; although they are calculated differently, these terms are used interchangeably in the fitness community as calculations yield approximately the same values.

actual multiplier will likely be lower than 1.2, but higher than 1; despite removing DIT from the equation, PAEE cannot be fully excluded as a sedentary person will still walk and fidget. Because DIT is estimated to account for 5 to 15% of TDEE depending on the composition of the diet (how much protein, carbohydrates etc.), this essay will assume $TDEE = 1.1 \times BMR$ for a person who eats nothing.

$$\frac{dW}{dt} = -1.1k \times BMR$$

We must also assign a value to k; the units of the right hand side of the differential equation is energy in calories, whilst the left is in kilograms. Therefore, k helps us view the calories expended as weight loss. The famous fitness aphorism of "3,500 calories per pound" (7700 calories per kilogram) only applies if all the weight lost was fat,[3] which is unrealistic as fat-free mass (FFM) will also be lost in a calorie deficit. An alternative is to use the quarter FFM rule; assuming weight loss is done through changes in diet alone, 75% of weight lost is fat, whilst 25% is FFM. If this rule holds true, then 7400 calories per kilogram is a more precise value to utilize. Hence, assume $k = \frac{1}{7400}$.[4]

$$\frac{dW}{dt} = -\frac{1.1}{7400} \times BMR$$

Substituting the Mifflin-St Jeor Equation for BMR (Men):

$$\frac{dW}{dt} = -\frac{11}{7400}W - \frac{1.1}{7400}(6.25H - 5Y + 5)$$

From here, there are two ways to solve this problem in two different ways. First is a numerical method (Euler's method), which can be done on a spreadsheet.

[3] By now it should be noted that it is technically mass that is being lost, not weight; however, we shall continue to refer to weight in kilograms as "weight loss" is more commonly used in everyday discussion than "mass loss."
[4] In a way, we are performing a sort of Fermi estimation as we assume overestimations and underestimations approximately cancel each other out. For example, we have already estimated the multiplier to convert BMR to TDEE, as well as the proportionality constant k.

Day	dW/dt	Weight/W (kg)	Height/H (cm)	Age/Y (Years)
1	-0.4392567568	180	200	20
2	-0.4386017712	179.5607432	200	20.00273973
3	-0.4379477594	179.1221415	200	20.00547945
4	-0.4372947196	178.6841937	200	20.00821918
5	-0.4366426507	178.246899	200	20.0109589
6	-0.435991551	177.8102563	200	20.01369863
7	-0.4353414192	177.3742648	200	20.01643836
8	-0.4346922537	176.9389234	200	20.01917808
9	-0.4340440533	176.5042311	200	20.02191781
10	-0.4333968164	176.0701871	200	20.02465753
11	-0.4327505416	175.6367902	200	20.02739726
12	-0.4321052275	175.2040397	200	20.03013699
13	-0.4314608726	174.7719345	200	20.03287671
14	-0.4308174756	174.3404736	200	20.03561644
15	-0.4301750349	173.9096561	200	20.03835616
16	-0.4295335493	173.4794811	200	20.04109589
17	-0.4288930172	173.0499475	200	20.04383562

Day	dW/dt	Weight/W (kg)	Height/H (cm)	Age/Y (Years)
184	-0.3342464023	109.6073556	200	20.50136986
185	-0.3337475133	109.2731092	200	20.50410959
186	-0.3332493658	108.9393617	200	20.50684932
187	-0.3327519588	108.6061123	200	20.50958904
188	-0.3322552913	108.2733603	200	20.51232877
189	-0.331759362	107.941105	200	20.51506849
190	-0.3312641699	107.6093457	200	20.51780822
191	-0.3307697139	107.2780815	200	20.52054795
192	-0.3302759929	106.9473118	200	20.52328767
193	-0.3297830058	106.6170358	200	20.5260274
194	-0.3292907516	106.2872528	200	20.52876712
195	-0.328799229	105.957962	200	20.53150685
196	-0.3283084371	105.6291628	200	20.53424658
197	-0.3278183748	105.3008544	200	20.5369863
198	-0.3273290409	104.973036	200	20.53972603
199	-0.3268404344	104.645707	200	20.54246575
200	-0.3263525543	104.3188665	200	20.54520548

Figure 2: Screenshots taken from spreadsheet

The numerical method works by calculating $\frac{dW}{dt}$ on day 1, knowing the initial weight, height and age. As $\frac{dW}{dt}$ is the change in weight across a day, we can add that to the day 1 weight to obtain the day 2 weight. Intuitively, the numbers seem correct since we have established the weight loss should be the greatest at the start (slightly under half a kilogram, or about a pound a day), and it tapers off to roughly 0.33 kilograms by the last day. At the end of the experiment, the man's weight falls from 180kg to roughly 104kg.

The advantage of this method is that it is unnecessary to solve the differential equation. Even if we had not made gross oversimplifications such as assuming calorie intake and DIT to be equal to zero, we can still calculate weight loss day-by-day, and even account for variations in activity levels from each day. Moreover, the step size can be varied to obtain an even more precise calculation; the units of time can be changed from days to hours or minutes.

The second method is to solve the differential equation exactly, also known as an analytic method.

$$\frac{dW}{dt} = -\frac{11}{7400}W - \frac{1.1}{7400}(6.25H - 5Y + 5)$$

$$\frac{dW}{dt} = \; - \frac{11}{7400}W + \frac{5.5}{7400}Y - \frac{1.1}{7400}(6.25H + 5)$$

$$\frac{dW}{dt} + \frac{11}{7400}W = \frac{5.5}{7400}Y - \frac{1.1}{7400}(6.25H + 5)$$

The equation now resembles the form $\frac{dy}{dx} + P(x)y = Q(x)$. Utilize the integrating factor:

$$I(x) = e^{\int \frac{11}{7400}dt} = e^{\frac{11t}{7400}+c} = Ae^{\frac{11t}{7400}}$$

(where A is a constant)

Multiplying both sides of the differential equation by I(x):

$$Ae^{\frac{11t}{7400}}(\frac{dW}{dt} + \frac{11}{7400}W) = Ae^{\frac{11t}{7400}}(\frac{5.5}{7400}Y - \frac{1.1}{7400}(6.25H + 5))$$

This simplifies to:

$$\frac{d}{dt}(Ae^{\frac{11t}{7400}} \times W) = Ae^{\frac{11t}{7400}}(\frac{5.5}{7400}Y - \frac{1.1}{7400}(6.25H + 5))$$

Integrating both sides with respect to t:

$$Ae^{\frac{11t}{7400}} \times W = \int Ae^{\frac{1.1t}{7400}}(\frac{5.5}{7400}Y - \frac{1.1}{7400}(6.25H + 5))\,dt$$

Y (age in years) is a function of time; let initial age be Y_i hence we shall substitute $Y = Y_i + \frac{t}{365}$:

$$Ae^{\frac{11t}{7400}} \times W = \int Ae^{\frac{11t}{7400}}(\frac{5.5}{7400}(Y_i + \frac{t}{365}) - \frac{1.1}{7400}(6.25H + 5))\,dt$$

$$Ae^{\frac{11t}{7400}} \times W = \int Ae^{\frac{11t}{7400}}(\frac{5.5}{7400}(\frac{t}{365}) - \frac{1.1}{7400}(6.25H - 5Y_i + 5))\,dt$$

$$Ae^{\frac{11t}{7400}} \times W = \int Ae^{\frac{11t}{7400}}(\frac{5.5}{7400}(\frac{t}{365})\,dt - \frac{1}{10}Ae^{\frac{11t}{7400}}(6.25H - 5Y_i + 5))$$

$$Ae^{\frac{11t}{7400}} \times W = A(\frac{5.5}{7400\times365})\int e^{\frac{11t}{7400}} t\,dt - \frac{1}{10}Ae^{\frac{11t}{7400}}(6.25H - 5Y_i + 5))$$

Solving for $\int e^{\frac{11t}{7400}} t\,dt$ using integration by parts:

$$u = t \quad du = dt$$
$$v = \frac{7400}{11}e^{\frac{11t}{7400}} \quad dv = e^{\frac{11t}{7400}}dt$$

$$\int e^{\frac{11t}{7400}} t\,dt = \frac{7400t}{11}e^{\frac{11t}{7400}} - \int \frac{7400}{11}e^{\frac{11t}{7400}}dt$$

$$\int e^{\frac{11t}{7400}} t\,dt = \frac{7400t}{11}e^{\frac{11t}{7400}} - (\frac{7400}{11})^2 e^{\frac{11t}{7400}} + c = e^{\frac{11t}{7400}}[\frac{7400t}{11} - (\frac{7400}{11})^2] + c$$

(where c is a constant)

Substituting once more to solve the original differential equation:

$$Ae^{\frac{1.1t}{7400}} \times W = A(\frac{5.5}{7400\times365})e^{\frac{1.1t}{7400}}[\frac{7400t}{11} - (\frac{7400}{11})^2] - (\frac{1.1}{7400})^2 Ae^{\frac{1.1t}{7400}}(6.25H - 95)) + c$$

$$e^{\frac{1.1t}{7400}} \times W = (\frac{5.5}{7400\times365})e^{\frac{11t}{7400}}[\frac{7400t}{11} - (\frac{7400}{11})^2] - \frac{1}{10}Ae^{\frac{11t}{7400}}(6.25H - 5Y_i + 5) + c$$

$$W(t) = (\frac{5.5}{7400\times365})[\frac{7400t}{11} - (\frac{7400}{11})^2] - \frac{1}{10}(6.25H - 5Y_i + 5) + ce^{\frac{-11t}{7400}}$$

To obtain c, simply substitute values we already know. At the start of day 1, $W = 180$ and $t = 0$:

$$180 = (\frac{5.5}{7400\times365})[-(\frac{7400}{11})^2] - \frac{1}{10}(6.25(200) - 5(20) + 5) + c$$

$$c = 296.4215 \approx 296$$
$$W(t) = (\frac{5.5}{7400\times365})[\frac{7400t}{11} - (\frac{7400}{11})^2] - \frac{1}{10}(6.25H - 5Y_i + 5) + 296e^{\frac{-11t}{7400}}$$

After 200 days, the final weight of the man will be:

$$W(200) = (\frac{5.5}{7400\times365})[\frac{7400(200)}{11} - (\frac{7400}{11})^2] - \frac{1}{10}(6.25(200) - 5(20) + 5) + 296e^{\frac{-11(200)}{7400}}$$

$$W(200) = 103.7280779 \approx 104kg$$

Recall that using Euler's method, we also obtained a weight of roughly 104kg at the end of the experiment. Although our imaginary man would not shed half his weight, 76kg is still a respectable amount![5]

Case Study: Angus Barbieri

Actually, a similar experiment to the above has been performed before. In 1965, 27 year-old Angus Barbieri fasted for 1 year. He went from 207kg to 82kg in the span of 382 days.

Figure 3: Barbieri before and after his 382 day fast

There are no reliable sources online listing his height. However, it is commonly known in mathematics that if you have an equation with one unknown variable, this variable can be found. Let's try to find his height!

As derived earlier:

$$W(t) = \left(\frac{5.5}{7400\times365}\right)\left[\frac{7400t}{11} - \left(\frac{7400}{11}\right)^2\right] - \frac{1}{10}(6.25H - 5Y_i + 5) + ce^{\frac{-11t}{7400}}$$

[5] Do not attempt to lose weight like this at home.

$$c = e^{\frac{11t}{7400}}[W(t) + (\frac{5.5}{7400\times365})[(\frac{7400}{11})^2 - \frac{7400t}{11}] + \frac{1}{10}(6.25H - 5(27) + 5)]$$

$$c = e^{\frac{11t}{7400}}[W(t) + (\frac{5.5}{7400\times365})[(\frac{7400}{11})^2 - \frac{7400t}{11}] + 0.625H - 13]$$

We have two frames of reference: $(W, t) = (207, 0)$ *or* $(82, 382)$. From this, we can obtain the following simultaneous equations.

$$Equation\ 1:$$
$$(W, t) = (207, 0)$$

$$c = 207 + (\frac{5.5}{7400\times365})(\frac{7400}{11})^2 + 0.625H - 13$$

$$c = 0.625H + 194.9215$$

$$Equation\ 2:$$
$$(W, t) = (82, 382)$$

$$c = e^{\frac{11(382)}{7400}}[82 + (\frac{5.5}{7400\times365})[(\frac{7400}{11})^2 - \frac{7400(382)}{11}] + 0.625H - 13]$$
$$c = 1.1028H + 122.4496$$

Setting Equation 1 = Equation 2:

$$0.625H + 194.9215 = 1.1028H + 122.4496$$

$$H = \frac{194.9215-122.4496}{1.1028-0.625} = 151.678cm = 152cm$$

Conclusion

It is hardly realistic to apply equations such as the Mifflin-St Jeor Equation to precisely predict one's weight loss progress; our bodies are idiosyncratic and weight loss depends on countless factors which may very well be impossible to quantify into one equation.[6] The few sources online available state that Barbieri's height go no lower than 170cm – which, despite their inaccuracy – suggests that the Mifflin-St Jeor Equation may fail in atypical conditions such as complete starvation or extreme obesity.

[6] Moreover, this essay only discusses weight loss rather than fat loss, which is usually more desirable.

Possible Extensions

I had a lot of fun writing this essay, but utilizing mathematics to model weight loss more comprehensively entails too many variables to cover in 2,000 words. In the future, it would be interesting to cover other weight loss models such as the Katch-McArdle, which models BMR as linearly related to FFM, or to analyze fat loss rather than weight loss.

References

https://www.cambridge.org/core/journals/journal-of-nutritional-science/article/testing-a-proposed-mathematical-model-of-weight-loss-in-women-enrolled-on-a-commercial-weightloss-programme-the-lighterlife-study/16D465BDCCCFA52B617B3CC35F1BBADE

https://pmc.ncbi.nlm.nih.gov/articles/PMC524030/

https://onlinelibrary.wiley.com/doi/full/10.1111/j.1467-789X.2011.00873.x