The Million Dollar Question: Understanding the Riemann Hypothesis

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1. Introduction

What if a single mathematical insight could be worth one million dollars?

The Riemann Hypothesis is one of the most famous unsolved problems in mathematics. In 2000 it was deemed by the Clay Mathematics Institute as one of the seven most challenging unsolved mathematical problems, being selected as one of the Millennium Prize Problems with a bounty of $1 million. However, it is regarded as such a difficult problem that most mathematicians will tell you there are far easier ways to make $1 million. So, what actually is the Riemann Hypothesis, how does it connect to prime numbers, and how did Bernhard Riemann's dabblings in complex analysis give us the world's greatest number theory problem?

2. Prime Numbers

Prime numbers, numbers which have no factors except 1 and themselves, are the building blocks of mathematics. The fundamental theorem of arithmetic states that every integer greater than 1 is either a prime itself or can be expressed in only one combination as a unique product of primes, and in this way they act much like the DNA of integers. This property of primes makes them central to number theory and the foundation of countless proofs, and as a result mathematicians have spent centuries trying to learn more about them and their distribution.

Since the work of Euclid in Ancient Greece it has been known that there exist infinitely many prime numbers. In 1792, then 15-year-old mathematician Carl Friedrich Gauss began the task of counting prime numbers up to 3 million in order to study their distribution. His findings led him to develop what is now known as the Prime Number Theorem:

$$\pi(x) \approx \frac{x}{\ln(x)}$$

where $\pi(x)$, the prime-counting function, represents the total number of primes less than or equal to x. Specifically, as x approaches infinity, the ratio of $\pi(x)$ to $\frac{x}{\ln(x)}$ approaches 1.

Later work building upon Gauss's research found that a more accurate approximation for the prime-counting function was the logarithmic integral:

$$\mathrm{Li}(x) = \int_2^x \frac{dt}{\ln t}$$

This, at the time, was the most accurate way to calculate the distribution of prime numbers.

3. The Zeta Function

In the 18th century, Leonhard Euler studied a function, later notated by Riemann with the Greek letter zeta (ζ), defined as:

$$\zeta(s) = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \frac{1}{4^s} + \cdots$$

Or, alternatively:

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

In 1734 Euler solved the case of $s = 2$ (i.e. the sum of the reciprocals of the squares of natural numbers), known as the Basel problem, proving the series converges to $\frac{\pi^2}{6}$. Euler also proved the function converges for all $s > 1$. In 1737, Euler established a relationship between the zeta function and the prime numbers, proving that

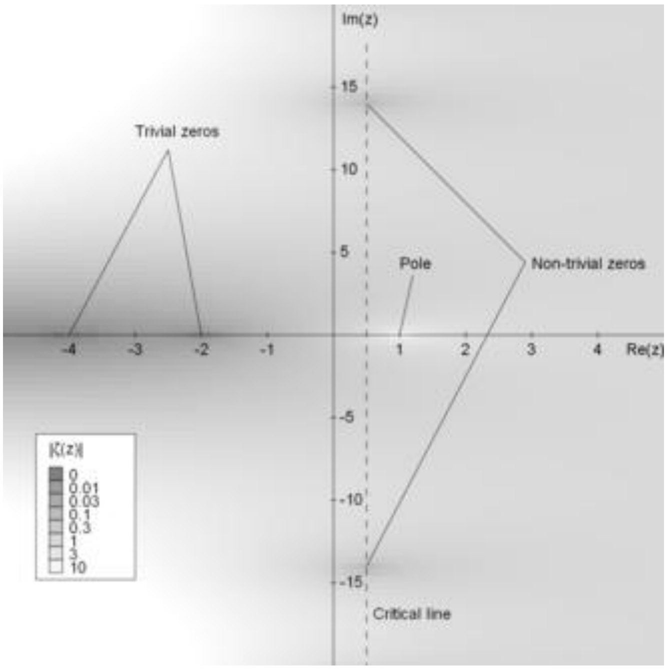
$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

where the infinite product extends over all prime numbers p.

In 1859, German mathematician Bernhard Riemann published his paper titled 'On the Number of Primes Less Than a Given Magnitude'. In the paper, he explored applying the zeta function to complex numbers. This produces a convergent result for all inputs where $\mathrm{Re}(s) > 1$. To expand this domain to all complex numbers, Riemann used a method called analytic continuation.[1] This resulted in the zeta function being defined for all complex $s \neq 1$. With this extension, Riemann was able to explore the behaviour of the function more deeply, and this is where something interesting and unexpected emerged.

At this point, you would be right to point out that the title of Riemann's paper relates to the distribution of prime numbers, but none of the analysis done so far seems to be remotely linked. Yet remarkably, this function encodes information about every prime number, hidden within its structure. Here is where a key aspect of the function comes in: the location of its zeros.

3.1 Zeros



A plot visualising the zeta function, with trivial and non-trivial zeros labelled, as well as the critical line at $\mathrm{Re}(s) = \frac{1}{2}$ and the pole at $s = 1$ (source: https://en.wikipedia.org/wiki/Riemann_hypothesis)

The extension revealed a striking and unexpected feature of the function regarding the placement of so-called 'zeros' – values of s for which the output of the zeta function is 0. These zeros are split into two categories: 'trivial' zeros and 'non-trivial' zeros. The trivial zeros occur at all negative even integers, as shown in the graph above. These zeros are called 'trivial' as their occurrence can be predicted and explained – the $\sin\left(\frac{\pi s}{2}\right)$ term in the functional equation equals 0 at these values for s. The non-trivial zeros, however, are harder to predict.

It is known that all non-trivial zeros must lie within the region $0 < \mathrm{Re}(s) < 1$.[2] This is known as the 'critical strip'. It has also been shown that the non-trivial zeros are symmetric about the real axis (i.e. if $\sigma + it$ is a zero, then $\sigma - it$ must also be a zero). Riemann proved there are infinitely many non-trivial zeros in this critical strip.

In his 1859 paper, Riemann observed that all of his non-trivial zeros lay on what he described as the 'critical line' – the line where $\mathrm{Re}(s) = \frac{1}{2}$. He hypothesised that all non-trivial zeros lie on this line. This is the hypothesis which now bears his name and a $1 million prize.

3.2 Relevance of the Hypothesis

But why does the placement of these non-trivial zeros actually matter? Riemann's paper was interested in a prime-counting function, much like Gauss's function. Riemann amended

Gauss's method of counting primes. Rather than jumping by 1 for every prime, his function $\Pi_0(x)$ had jumps of $\frac{1}{n}$ at prime powers p^n. This gave a closer approximation of how primes are distributed, but not a perfect one. However, by subtracting a sum over the non-trivial zeros of his zeta function, he produced a formula which exactly expressed the prime-counting function $\pi(x)$. In other words, knowledge of the location of non-trivial zeros of the zeta function can tell us the exact distribution of prime numbers. These zeros are used in the Chebyshev function[3], a more accurate prime-counting function than those which came before and one often used in proofs related to prime numbers. If the Riemann Hypothesis is true, this would imply that the error margin in these prime-counting formulas is as small as possible, giving us the most accurate estimates of the distribution of prime numbers. It would also imply a regular and predictable distribution of prime numbers, suggesting a deeper underlying structure beneath their apparent randomness.

Proof of the hypothesis would also validate a large amount of existing mathematical work that has been proved contingent on this result, while any proof would likely lead to many new ideas and techniques, opening up plenty more avenues of research within number theory and beyond. A proof would have practical applications particularly in the field of cryptography, where many modern encryption algorithms rely on primes, as well as other branches of maths and physics.

This is all wonderful, but for the slight hindrance that the hypothesis remains merely a hypothesis. Over 10 trillion zeros have been checked computationally, with every single one following the hypothesis, but there are infinitely many zeros and so proof by this method is impossible. As already mentioned, this is regarded as one of the hardest problems currently in mathematics, with Terence Tao claiming simply that "the tools are not there" to solve it. Far more breakthroughs and discoveries would likely have to be made before any successful attempt to solve the hypothesis could begin. A good way to make a million dollars, but by no means an easy one – the Clay Mathematics Institute aren't likely to be paying out any time soon.

4. Patterns in Mathematics

While it is unlikely we solve the hypothesis any time soon, there is still much thinking to be taken from it. It is striking, to me at least, how unrelated fields of mathematics can come together to form such seemingly beautiful and ordered patterns, and even more fascinating that despite mathematics being the language we have invented to describe patterns like these in the universe, it is such a struggle to apply that language to such apparently simple occurrences and patterns. In this claim I have touched upon another question of mathematics: are these patterns discovered or have we as humans invented them? I personally feel that these patterns are inherent in nature, and emerge from the essence of these natural mathematical concepts, much in the same way that Fibonacci sequences emerge in petals, fractals in snowflakes and logarithmic spirals in galaxies. Mathematics to me is the constructed language we use to transcribe and make sense of these natural phenomena, and it is this language which we use to aid our quest for discovery. So for now, while we may not be able to solve problems such as the Riemann Hypothesis, they still represent the beauty of mathematics, the fundamental building blocks of number theory coming together with a simply-defined function in the complex domain to form natural, inherent patterns of unexpected regularity.

5. Conclusion

The Riemann Hypothesis is an almost 170-year-old problem which has thus far eluded mathematicians, despite countless attempts ranging from proofs to brute force to cheap shots at a million dollars. Arguably the greatest unsolved problem in number theory, this simple hypothesis may well be hiding all we could ever wish to know about the very building blocks of mathematics. Most mathematicians believe it to be true, and over 10^{13} zeros have been checked, yet none of these can definitively describe an infinite pattern except a proper mathematical proof. If we can't solve it just yet, we can still reflect on the questions it raises, and the beauty of patterns we do not yet fully understand. Perhaps even within the most fundamental elements of mathematics, there remains far more to discover. But for now, the Riemann Hypothesis remains unsolved – a million-dollar question at the heart of mathematics.

6. Appendix

1. The analytic continuation of the zeta function for $0 < \mathrm{Re}(s) \leq 1, s \neq 1$ makes use of the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}$$

by defining

$$\zeta(s) = \frac{\eta(s)}{1 - 2^{1-s}}$$

for $0 < \mathrm{Re}(s) \leq 1$.

The analytic continuation for $\mathrm{Re}(s) \leq 0$ makes use of the functional equation:

$$\zeta(s) = 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \zeta(1-s)$$

where $\Gamma(s)$ is the gamma function:

$$\Gamma(s) = \int_0^{\infty} x^{s-1} \, e^{-x} dx$$

2. This is due to the function's convergence for $\mathrm{Re}(s) > 1$ and the functional equation only allowing for trivial zeros for $\mathrm{Re}(s) < 0$. It may also be proved that non-trivial zeros cannot lie on the lines $\mathrm{Re}(s) = 0$ or $\mathrm{Re}(s) = 1$.

3. The Chebyshev function is defined as:

$$\psi(x) = x - \sum_{\rho} \frac{x^{\rho}}{\rho} - \ln(2\pi) - \frac{1}{2} \ln(1 - x^{-2})$$

where ρ are the non-trivial zeros of the zeta function.