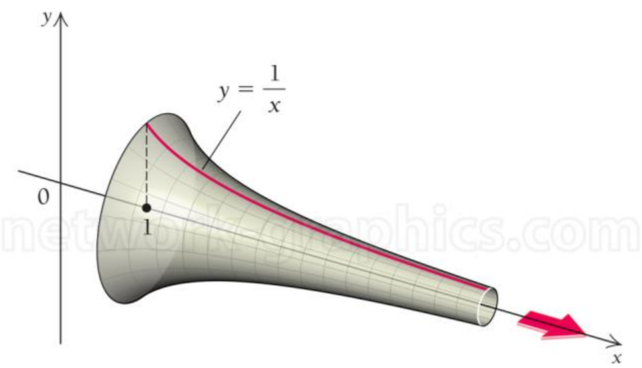
The Mystery of Infinity

Infinity is defined as something boundless, endless, or without limit. It is something we find incredibly difficult to visualise. Usually, when we try to imagine things, we compare them to things we are already aware of. However, infinity is not a concept to which we can easily draw parallels, as there are very few things, if any, that are boundless, endless or without limit. Even if we look into the horizon or out at sea, which appears to stretch on forever, we know that it does not; it eventually comes to an end. This is why I found it so fascinating to discover cases of infinity appearing in geometry. How could something endless appear in geometry, which is the study of shapes, sizes and the properties of space. How can something boundless exist within space? Does this mean that space itself is infinite?

Gabriel's Horn is named after the archangel Gabriel who is believed to blow a horn to announce judgement day. The horn is created by rotating the graph y = $1/x$ around the x-axis from x = 1 to x = ∞. This can be seen on the graph below.



This intriguing shape was discovered by Evangelista Torricelli, an Italian physicist, mathematician and student of Galileo Galilei, in 1641. You might assume that because the end of the horn goes on to infinity, both the volume and the surface area of the horn is infinite. This logically makes sense; however, this is not the case. The volume of the horn is finite (meaning it has an end) while the surface area is infinite. This means that this horn could be filled to the brim with paint however it is impossible to paint the inside and outside of the horn. Therefore this problem is known as the Painter's Paradox.

Imagine you wanted to find the volume of the horn. You could cut it up into an infinite amount of extremely thin slices across its cross section. Then you could add up the areas of all these slices to get the volume of the horn. The cross section of the horn is a circle so the equation for the area is πr^2. The radius of the cross section is the y-value at that point. Because y = $1/x$ then the $r = 1/x$. So a general formula for the area:

$$\text{area} = \pi r^2$$
$$\text{area} = \pi\left(\frac{1}{x}\right)^2$$
$$= \pi\left(\frac{1}{x^2}\right)$$
$$= \pi x^{-2}$$

Because we have cut the horn into extremely thin slices we can use integration to calculate the volume.

$\int_1^\infty \pi x^{-2}\,dx \rightarrow$ this means we add up all the slices of crossectional area from 1 to ∞

Because π is a constant, we can take it out of the integral:

$$\pi \int_1^\infty x^{-2}\,dx$$

Because infinity is a concept not a number, we can't integrate it. What we can do is integrate b as it gets to infinity.

$$\pi \int_1^\infty x^{-2}\,dx$$
$$= \pi \times \lim_{b \to \infty} \int_1^b x^{-2}\,dx$$

After evaluating this integral we get:

$$= \pi \times \lim_{b \to} \left[-\frac{1}{x}\right]_1^b$$

as b gets bigger and bigger, $\frac{1}{b}$ gets smaller and smaller

so as $b \to \infty$, $\frac{1}{b} \to 0$

this means we can write $\frac{1}{b} = 0$

when $x = 1$, $\frac{-1}{x} = \frac{-1}{1} = -1$

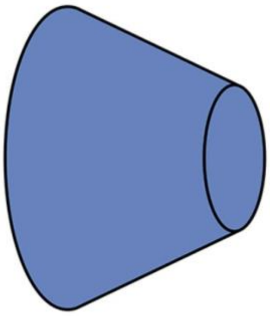
so $\pi\left(\frac{-1}{b} - \left(\frac{-1}{1}\right)\right)$

$= \pi(0 - (-1))$

$= \pi(1)$

$= \pi \rightarrow$ the volume $= \pi$ which is a finite number

In order to calculate the surface area, we can imagine the horn as a frustum which can be seen below.



Using this shape, and a lot of algebra, (which Tom Crawford explains wonderfully), we end with a formula for the surface area of the horn:

$$S.A = 2\pi \int y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \; dx$$

As seen in the workings out above, y = 1/x. This means that:

$$\frac{dy}{dx} = -\frac{1}{x^2}$$

We can put these values into the integral and we end up with

$$S.A = 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \left(\frac{-1}{x^2}\right)^2} \; dx$$

$$= 2\pi \int_1^\infty \frac{1}{x} \sqrt{1 + \frac{1}{x^4}} \; dx$$

Because this integral contains a square root, it is not as simple as what we did previously when finding the volume. What we can do is use comparison to help solve this problem. We know that the answer to the square root is greater than one. This

is because any value of x, will give an answer of 1 point something which is still greater than 1.

$$\sqrt{1 + \frac{1}{x}^4} \geqslant 1 \quad \text{and} \quad \frac{1}{x} > 0$$

$$\text{so} \quad \frac{1}{x}\sqrt{1 + \frac{1}{x^4}} \geqslant \frac{1}{x}$$

We can compare the integrals of both sides of this inequality.

$$\text{if } \frac{1}{x}\sqrt{1+\frac{1}{x^4}} \geqslant \frac{1}{x}$$

$$\text{then } 2\pi \lim_{b\to\infty}\int_1^b \frac{1}{x}\sqrt{1+\frac{1}{x^4}}\,dx \geqslant 2\pi \times \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$

$$\text{so } S.A \geqslant 2\pi \times \lim_{b\to\infty}\int_1^b \frac{1}{x}\,dx$$

If we evaluate the right hand side of the inequality:

$$2\pi \times \lim_{b\to\infty}\int_1^b \left(\frac{1}{x}\right)dx$$

$$= 2\pi \times \lim_{b\to\infty}\left[\ln(x)\right]_1^b$$

$$= 2\pi \times \infty$$

$$= \infty$$

Then we are able find the surface area:

$$\text{so } S.A \geqslant \infty$$

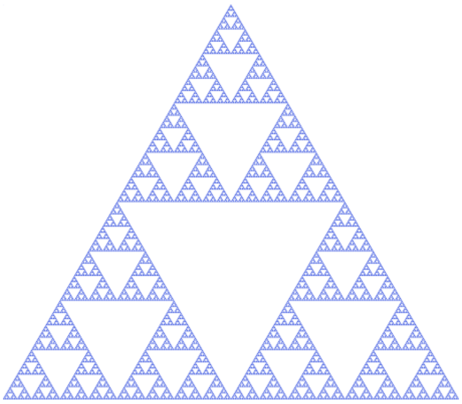
$$S.A = \infty$$

This is because if something is greater than infinity , then it too must also be infinite. Therefore the surface area of Gabriels Horn is infinite, while the volume is finite.

After coming to this conclusion I was left with more questions then answers. If the horn goes on forever, how are you ever able to fill it up? Surely the paint would continue to leak out of the other side of the cone for infinity? However, because the other end of the horn goes on for infinity, there is no exit point so the paint cannot leak out!

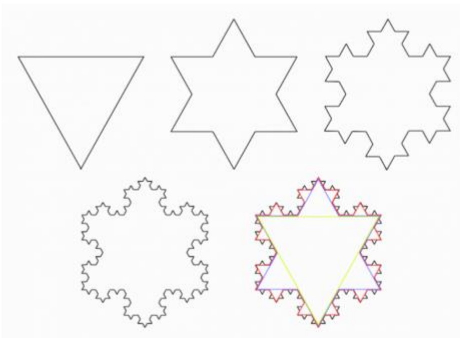
If the cone is filled up with paint then wouldn't this also paint the inside of the cone? However, the key idea about this horn is that it exists in a mathematical realm where things happen differently than in 'real life'. Volume and surface area measure fundamentally different things. Surface area relates to 2D, while volume relates to 3D. Therefore, we cannot really compare what happens in these different dimensions. This means that filling something in 3D is different from coating something in 2D. We could fill the 3D aspect of the horn because it is finite, but the 2D aspect (the surface area) never gets coated because it is infinite. This reveals the multifaceted nature of infinity as it behaves differently depending on how it is measured. This paradox is fascinating because it completely breaks away from reality, highlighting how complex and exciting infinity is.

Another example of infinity in geometry is found in fractals. According to the Fractal Foundation, “A fractal is a never-ending pattern. Fractals are infinitely complex patterns that are self similar across different scales.” What this means, is that no matter how many times you zoom into the fractal, it still looks exactly the same. One example of a fractal is called the Sierpinski Triangle.



The pattern starts with an equilateral triangle. Then an upside down equilateral triangle is placed in the middle of the original triangle which splits it into three identical triangles. Then we repeat this process in each of the new triangles and this goes on for infinity. Each smaller triangle is identical to the original triangle, and so this pattern is self similar.

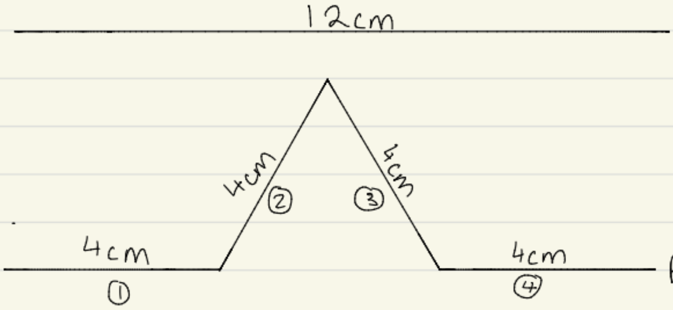
Another fascinating example of a fractal is Koch’s Snowflake. It is named after the Swedish mathematician Helge von Koch, who created the shape in 1904. It is made by dividing the sides of an equilateral triangle into thirds with length x. The middle part is removed and replaced with two sides, each, still with length x which creates a new equilateral triangle on the side of the original triangle. We repeat this process for infinity and each new triangle is identical to the original triangle showing that this shape is also self-similar. This can be seen below:



The reason why this particular fractal is so interesting is because it is another example where infinity in geometry appears counter intuitive. The perimeter of this shape is infinite however the area inside is finite. The area being finite is fairly intuitive as it is an enclosed shape. The perimeter is slightly more interesting.

* In each step we increase the number of sides $\times 4$
* And we decrease the length of each side $\times \frac{1}{3}$
* So the perimeter increases by $4 \times \frac{1}{3} = \frac{4}{3}$

e.g



no. of sides = 1
perimeter = 12cm

no. of sides = 4
perimeter = 4+4+4+4 = 16 cm

scale factor : $\frac{16}{12} = \frac{4}{3}$

so after n steps, where $x = P_0$ (Perimiter at 0 steps)

$$P_n = x\left(\frac{4}{3}\right)^n$$

since $\frac{4}{3} > 1$, the perimeter increases after each step
so as $n \to \infty$, the perimeter keeps growing forever
this means that $p = \infty$

At first glance, saying you can have a shape with an infinite boundary but a finite area sounds impossible. However, Koch's Snowflake proves otherwise, showing that infinity forces us challenge our intuition and that maths doesn't always align with everyday experience. This challenge highlights the counterintuitive yet elegant nature of mathematics. A simple process, repeated infinitely, creates a pattern which is both complex and surprisingly beautiful.

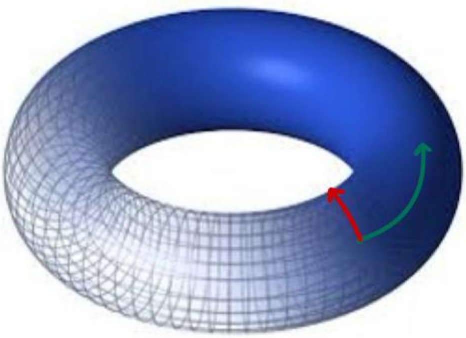
You might say that these examples of infinity only exist in a mathematical realm, but infinity is all around us. Infinity may define our universe.

We can divide the universe into two parts: the observable universe and the unobservable universe. Since the Big Bang, 13.8 billion years ago, which resulted in the universe, it has continued to expand, and so space itself has been stretched since its beginning. This means that we can see objects which are 46 billion light years away in every direction (which is around 435,000,000,000,000,000,000,000 km). The observable universe is absolutely massive.

Because light has had a limited amount of time to reach us, we cannot see further than this distance. Scientists have theories as to what lies beyond this. Some scientists suggest that the universe is infinite. Due to observations made by cosmologists, if the universe is infinite, then it is likely to be flat, almost like a sheet of paper. This would mean that space continues in all directions forever without a boundary or a limit. This creates the possibility of an infinite number of galaxies. Hypothetically, if there is an infinite amount of matter in the infinite universe, and if there is a finite number of possible combinations in which matter can be arranged, then a duplicate of our galaxy (the Milky Way) could exist. A duplicate of you or me could exist!

Some scientists propose that the universe is finite but without a boundary. While this sounds slightly confusing, we can use our own Earth as an example. The 2D surface of the Earth (which is a sphere) does not have a boundary. You could travel from point A to B without ever coming across a boundary, but the Earth is still finite. Similarly, the universe could be the 3D 'surface' of a 4D shape known as a Hypersphere, where space curves back on itself.

Other scientists suggest that the universe is curved, however, in a shape similar to a torus (which looks like a doughnut) which can be seen below.



This again shows that the universe could be boundless but finite. In this structure, theoretically, light could travel in a loop in two directions. It could go all the way round in the direction of the green arrow (which would take longer) or it could go in the direction of the red arrow (which is quicker). This means that you could see the same event happening twice at different times from the same spot!

I lean more towards the idea of the universe being infinite. The vastness of the observable universe leads me to believe that what is out there must be much more magnificent; infinite magnificence.

In conclusion, the examples of infinity in geometry through Gabriel's Horn and Koch's Snowflake show how progress in mathematics has allowed us to describe and work with the infinite. Despite this, I don't think it is a concept that we will ever fully comprehend. However, the ability to acknowledge these limits, I think, is the true measure of human intelligence. While we are able to appreciate its complexity, infinity forces us to be intellectually humble. I think it is rather freeing to be reminded of just how insignificant we truly are in the grand scheme of the wider universe. A key takeaway from this essay is that despite thousands of years of advancements in mathematics and science, there is still so much more we do not know. It is more plausible that the journey of understanding infinity will remain an infinite adventure.

References:

https://youtu.be/isdLel273rQ?si=4N7NgPjn56n72s53

https://youtu.be/yZOi9HH5ueU?si=4Ooy8ifnjDXIzn7V

https://youtu.be/3WVpOXUXNXQ?si=My1NCieRCl-mf31j

https://brilliant.org/wiki/gabriels-horn/

https://math.jhu.edu/~brown/courses/s12/ExtraProblems/Section7.8Problem.pdf

https://www.esa.int/Science_Exploration/Space_Science/Is_the_Universe_finite_or_infinite_An_interview_with_Joseph_Silk

https://geomaths.co.uk/2020/03/08/gabriels-horn/

https://www.skyatnightmagazine.com/space-science/how-big-universe

Gabriel's Horn Diagram: https://www.network-graphics.com/3d-surface-with-mesh-color-and-specular-highlights/

Frustum Image: https://mammothmemory.net/maths/geometry/circumference-area-and-volume/frustums.html

Torus Image: https://simple.wikipedia.org/wiki/Torus#/media/File:Torus.png

Sierpinski Triangle Image: https://en.wikipedia.org/wiki/Sierpiński_triangle

Koch's Snowflake Diagram: https://nrich.maths.org/problems/koch-snowflake