The Question Nobody Asked: On the Late Arrival of Calculus

By Fraser Mathew, Aged 13

Calculus is a mathematical framework which underpins much of modern physics and mathematics. But in the long history of these fields, calculus is only a recent invention. Trigonometry and algebra were developed much earlier. But calculus wasn't absent for lack of trying. Thinkers from Zeno to René Descartes grappled with questions about change and motion. Yet calculus wasn't invented until the middle of the seventeenth century. Why only then did calculus arrive? Was it a problem of tools, or was it a problem of philosophical ideas about dynamics and motion, what calculus describes?

Ancient Greece

To understand why calculus didn't come to fruition earlier, we can begin in Ancient Greece, where the first ideas which would become core elements of calculus emerged. Archimedes used inscribed polygons to approximate curves, and understood that the more sides of the polygon, the better the approximation. Other Greek geometers such as Euclid and Apollonius of Perga developed the idea of the tangent line.

But although their ideas were similar to those expressed in modern calculus, because Greek mathematics was primarily geometric, they didn't have a formal or general mathematical framework for expressing limits or continuous change.

To illustrate this, below I have displayed a method to find the tangent line, and thus derivative at any point of a parabola. However when generalizing attempts are made, we see a clear limitation of geometric interpretations

1. Construct a locus of points equidistant from the directrix and the focus. A traditional parabola, as defined by the Ancient Greek.
2. Construct a chord. A chord is defined as the secant line between the two points of intersection on a shape.

3. Construct a diameter of the parabola which bisects the chord. A diameter of a parabola defined by the Ancient Greeks as any line which uniquely intersects the parabola and is parallel to the axis of symmetry.

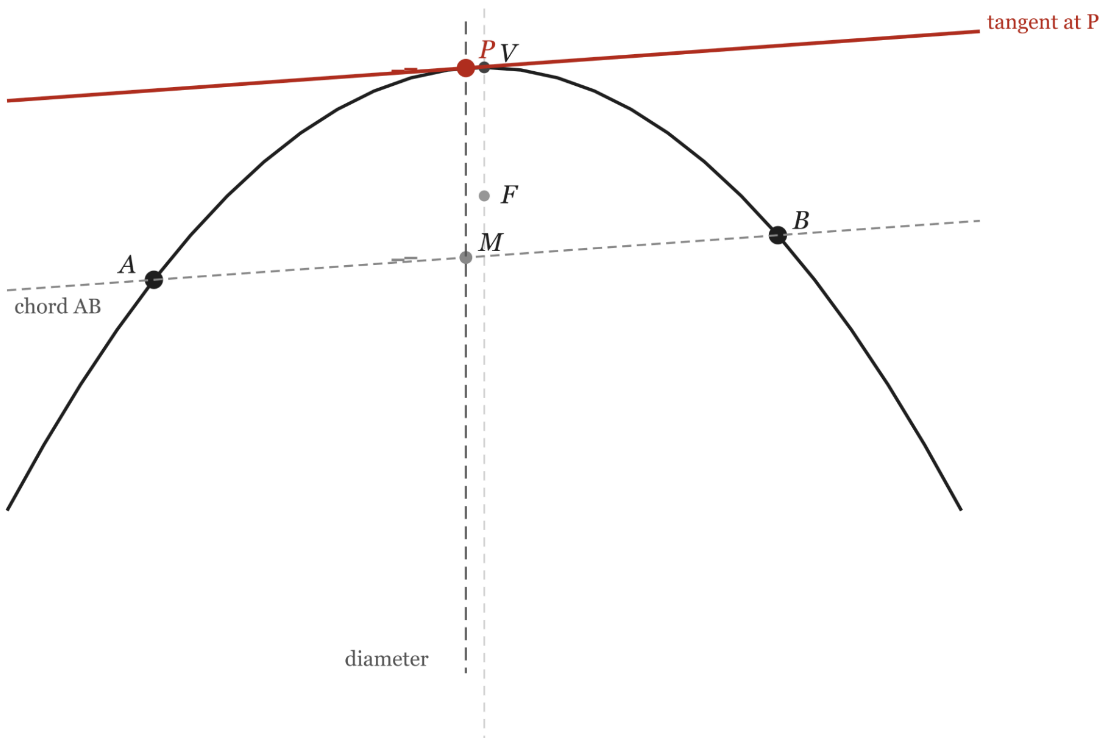

Fig. 1 — The diameter through M bisects chord AB and meets the parabola at P, where the tangent at P (red) is parallel to AB. After Apollonius, Conics I.46.

The construction above is one I myself created, but a proposition from Apollonius Of Perga's Conics, volume 1, proposition 46 helps prove an important relationship. This relationship, when put in these terms resembles the modern derivative, though limited to the parabola. This proposition proves that any diameter of a parabola which bisects a series of parallel chords, must intersect the parabola at a point where the tangent is parallel to the chords. In modern terms this means the slope of the tangent line must equal that of the chords the diameter of the parabola intersects. I shall link the pdf of it at the bottom of my essay, in case you wish to read further on this proposition.

This proof works very well for a parabola, and, indeed, similar proofs to do with other mathematical shapes could help us explore the derivative of them. But a problem arises. The Ancient Greeks thought in a pure geometrical sense. Because these methods depended on the specific qualities of each curve, this method did not naturally extend to arbitrary curves in a generalizable way. But was this only a limitation of their mathematical framework? Maybe, as they never truly tapped into dynamically describing curves in this sense, does this point to a deeper, conceptual limitation?

So is this the reason it took calculus so long to appear? The tools humans needed weren't fully developed yet, or not viewed in a sense which would make a framework like calculus possible? It is not the only one however, nor was it the most important, especially in later years to come.

A Problem Of Philosophy

Ancient Greek Mathematicians dealt mainly with planes where they could construct perfect circles and straight lines. They generally prioritized exact geometric forms over more abstract or inexact representations. One reason Greek geometers didn't push into dynamics, nor really explore the physical world viewed quantitatively was a problem of philosophy.

Aristotle created ideas about dynamics, and how things in our world interact, but he, as did almost every Greek thinker and mathematician, only viewed such ideas through a purely philosophical lens. For the Greeks, mathematics was about eternal, static truths. Perfect forms. A framework for describing change would have been philosophically uncomfortable, almost a contradiction within the way they viewed motion. The Greeks partially lacked the tools, but their world view, the way they chose to look at these problems, lacked the mathematical ideas and questions necessary. They made the distinction between natural motion and forced motion. A rock falling versus you picking it up and throwing it. But they did not develop a systematic mathematical framework for describing continuous change and motion. The way the rock fell off the cliff, how fast it fell, or at what rate it came toward the ground. Their philosophical lens meant their framework did not naturally prioritise the questions calculus was built to answer, damaging their ability to do so just as much as lacking the strict mathematical framework ever could.

And not only this. The Greek directly opposed the idea of infinity, with thinkers like Zeno of Elea creating paradoxes which made infinity seem unuseful and nonsensical. Ancient Greece birthed ideas about the curve, the approximation of it, and made many more advancements. Though their mathematical framework was flawed, their philosophical views prevented them from framing these problems in a way that led naturally to a general theory of change, because the ones they held directly opposed concepts of mathematical motion. Ideas required for calculus. But when would the need for a new philosophical view be met, and when would their physical tools allow them to elaborate?

The Golden Age of Islam

The Golden Age of Islam was a time period where new mathematical ideas about algebra, general summation, and series were developed rapidly. During this period, because of mathematicians like Al-Khwarizmi, often considered the father of algebra, Al-Karaji, and many other amazing thinkers, a better mathematical framework was created to work within. The Golden Age of Islam brought about modern algebra. Mathematicians created the ideas of polynomials, systematic ways to solve them, and even ways to find the maxima and minima. But calculus wasn't discovered during this period, even though many of the necessary ideas were uncovered.

To show this, let us begin with a function, something the mathematicians of The Golden Age of Islam knew well. Though I must note, while the algebraic tools available in this period were sufficient to approach this problem, Islamic mathematicians would have expressed this reasoning in words rather than symbols. The notation here is modern, used for clarity.

f(x) = x^2.

Next, using our knowledge of the slope of a line being:

$(y_2 - y_1) / (x_2 - x_1)$

we can plug this in for f(x), with the length of our line horizontally being a random variable, *h*.

$$\frac{f(x+h)-f(x)}{(x+h)-x}$$

$$\frac{(x+h)^2-x^2}{h}$$

$$\frac{x^2+2hx+h^2-x^2}{h}$$

$$\frac{2hx+h^2}{h}$$

$$2x+h$$

This simplifies to 2x + h, which represents the average rate of change of the interval, whose length is h. But as h gets smaller and smaller, that interval becomes incredibly minute, until the slope of the line whose length is that interval is almost equal to the slope of the parabola. In modern terms, this process relies on the idea of a *limit*: the value a function approaches as a variable moves closer and closer to a given point. In this case, as h approaches 0, the expression approaches a single value, which defines the exact slope of the curve at a point.This shows algebraic techniques were sufficiently advanced to approach problems resembling calculus. However, without a unifying conceptual framework about limits and continuous change, these tools were not yet organized into a general theory.

Why? They had many necessary tools at their disposal, did they not? Despite major advances in algebra and series, a general theory of continuous change did not emerge during this period. The broader mathematical culture was oriented toward algebra, astronomy, and geometry. These were powerful tools, but aimed mostly at static problems. Finding roots, describing shapes, calculating volumes. Islamic scholars often framed mathematical work as serving practical or theological ends: astronomy for prayer times, algebra for inheritance law, geometry for architecture. These were often static, discrete problems, generally looking past ideas like continuous change. Though ideas of motion were questioned, the concept of infinity as a framework itself wasn’t formalized. The question of continuous change wasn't uncomfortable to them the way infinity was for the Greeks. Infinity just wasn’t being explored as a method itself, and instead as an idea which could extend to better approximation. Ibn al-Haytham exemplifies this. He summed infinitely thin discs to find the volume of a paraboloid. Integral calculus in all but name. He even developed a systematic form for solving the area of a parabola of the form x to the n. But he never acknowledged what n itself represented as it climbed higher and higher. He never recognised the process itself as a general tool, even throughout his works in motion. Islamic scholars didn't directly oppose ideas like infinity and continuous motion like the Greeks, they just tended not to ask the questions.

The Scientific Revolution

The Renaissance changed the world immensely. Not just within its discoveries, but within its philosophy toward the world. The production and development of art and culture stemming from Florence brought with it questions, and not just that, a new philosophy. One which supported questioning the world's systems alongside religion, partially due to new ideas about the interpretation of the Bible. Unlike the Golden Age of Islam where these questions were unlikely to be posed, or Ancient Greece, where these ideas were directly opposed, the Renaissance actively made research like this prestigious.

One figure who embodied this shift was Galileo Galilei. In Two New Sciences, he formalised the relationship between space, time and acceleration, finally treating motion as something mathematics could describe rather than philosophy could merely categorise. He, along with the Scientific Revolution during the Renaissance, created a mass production of new scientific and mathematical ideas, helped normalise ideas surrounding movement as a mathematically tractable system. Not as an independent system, dictated by some inexpressible force. Galileo proved for instance, that the distance a falling body has traveled is equal to the square of time. He did this by measuring units time in relation to distance. He also found that there was a uniform force pulling everything towards the ground. These ideas, seeking mathematically defined answers to physical questions was something greek mathematicians would likely have refrained from due to their strict philosophical lens, and something, during the Golden Age of Islam that would rarely enter the investigative view of mathematics.

This wasn't the only philosophical change during this period. The Greeks had treated the infinite, as well as the infinitely small, with much caution. Ibn al-Haytham calculated the volume of a paraboloid using more and more discs, scratching the surface of integral calculus; he used infinity, but never formalized it into a mathematical framework, nor did any other mathematician truly, from that time period. But during the Scientific Revolution, a new, slight openness, due to pursuits in its favor from people like Cavalieri, a mathematician who used infinitesimals alongside traditional methods to help prove infinitesimals' validity, allowed people to experiment with the idea, and eventually, bring it to an accepted place within our mathematical world. Motion stopped being categorized as something to be viewed philosophically, but also as a genuine mathematical system. This allowed the stage to be set for a framework which requires both of these philosophical changes. Calculus.

Calculus's Emergence

As focus in mathematics shifted to the description of dynamics and of the world, it became clear that a method was needed to describe movement, independent of strict geometric rules. And though new tools existed by the time calculus was created, without the philosophy which had been born during the renaissance, these ideas would have been less useful. Yet without the right tools, even the correct questions would have gone unanswered. Descartes and Fermat particularly helped, using the shift of philosophy to allow them to create things like analytic geometry and the Cartesian plane. Newton first described this in the method of fluxions, which used time dependent variables, fluents, to describe movement. This was an innovation which used the idea of infinitesimals to describe continuous motion. Leibniz at a similar time created a similar method, which he named calculus, before publishing it, unaware of Newton's method of fluxions, leading to a bitter rivalry between the two. Though not enjoyable for either, this arrival at the same idea at the same time does show something however. Finally, the conditions for calculus were correct.

Calculus was ready, and became the framework we know today. But as an idea, calculus represents something exceedingly interesting. In hindsight, calculus could have emerged earlier. But it remained unformed. Calculus represents the idea that it isn't always mathematical prowess or random advancement which leads to discovery, but a reshaping in the way ideas are thought about. The question then is not whether we are clever enough. It is whether we are asking the right things at all within our endeavours.

Footnotes:

Apollonius of Perga, Conics, Book I, Proposition 46. [link to PDF]

Galileo Galilei, Two New Sciences (1638). Available at Internet Archive: [URL]

Ibn al-Haytham, On the Measurement of the Paraboloid. Discussion in The Origins of the Infinitesimal Calculus (1969), Ch. 1.

Margaret Baron, The Origins of the Infinitesimal Calculus (1969), Section 4.3 (Cavalieri).

Muslim Mathematicians - Zin Eddine Dadach [URL]