The shape which maximises area for a given perimeter can never be a Concave Shape.

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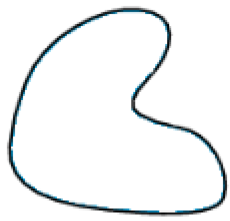
Introduction

This Essay shows that if we want to find the shape which maximises area for a given perimeter then that shape can never be a concave shape. As finding the shape which maximises area for a perimeter is a difficult enough task this simplifies that by eliminating all concave shapes, even though this essay won't show which shape maximises area for some perimeter, it is a nice problem on its own. So to show that we first identify the inward bulge.

Identifying the Inward Bulge

Consider a Concave closed shape as shown below

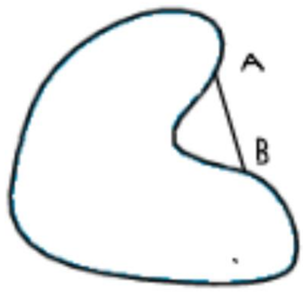


for any concave shape we can find two points inside the shape which when joined form a line which either partially or entirely lie outside the shape (this defining idea of a concave shape)

If we find a line which partly lies outside the shape, then we can find a point on the boundary and on the same line for which the whole line lies outside the shape, this is because the boundary separates the inside with the outside so to go from inside to outside the line must cross the boundary.

This means we can always find two points on the boundary of a concave shape such that the whole line lies outside.

Let's call those two points A and B (line AB lies outside the shape)



Earlier, we have stated that it is a closed continuous shape. But what does it mean to be continuous and closed?

If for any shape, we can go along the boundary and get to the same point we started from then the shape is closed. This is what we will use as the definition of a closed shape.

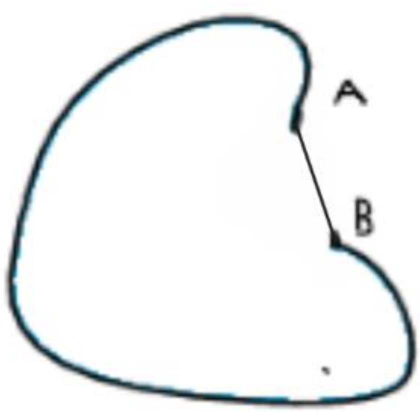
Then what about the shape being continuous? Now if a shape is continuous then we can always find points around a specific point on the shape for which no matter how small we make the region around that point we can always find points of the shape.

With that information we can continue forward.

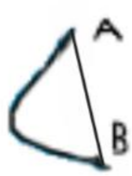
Now we know this shape is closed and continuous so, we can always get back to the same point we began form. But there are two paths it can take from a point, one would be upward and the other would be downwards (if at any point it does not point up or down we can turn it such that it does). As they go from the boundary it comes across all points on the boundary. So if we began with A then we can get to B from both paths (upward and downward).

Even though we can see which one is the bent in part, how can we definitively say that one is bent in and the other is not?

If we observe, then we notice that we can have two shapes formed:



And



(shape 1) (shape 2)

Now we consider the set of points inside the shape, then for each of the above shapes i.e. shape 1 and shape 2 let's call their sets of all points contained inside the shape (not including boundary) as S_{i1} and S_{i2} (i is for inside).

Then the intersect (the set of all elements common in the sets considered) of both the above sets with the set of all the points inside our original shape (the shape given beside) decides which part bulges in or out. The set of all points inside the original set shall be denoted as S_i.

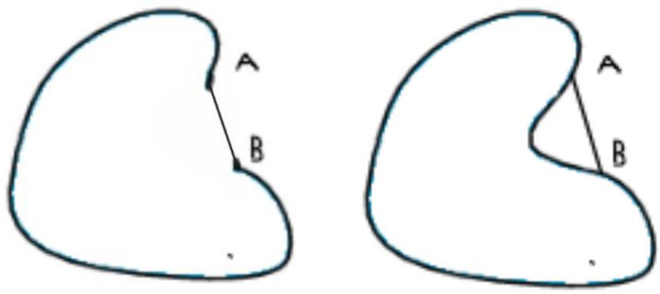
Then the bulged in part of the figure is the shape whose set containing all points inside it intersected with the set of all points inside the original shape is the empty set (in other words which shape does not share any common parts with the original shape).

i.e. if $S_{i2} \cap S_i = \Phi$ and $S_{i1} \cap S_i \neq \Phi$ then we can say that the shape whose set of all points inside the shape was S_{i2} (i.e. shape 2) would be the part bulge in wards.

Now that we have identified the bulged in part we can begin working on our main goal, to show that for any concave shape we can find shape with larger area but smaller perimeter.

Comparing the Bulge to the Chord

Consider shape 1 that we made earlier and our original shape.



Both have almost the same perimeter the only part which differs is the part between A and B, but which one is larger? (you may be saying, Idiot! The one which looks bigger is the one which has higher perimeter, but to show that we need to measure the length of the paths, which while considering every single possible inwards bulge, is an impossible task to measure them all.) Now we want to compare the concave part between A and B with the line segment AB.

So, for this task we are only interested in the bulged in part and the line segment AB. If we recall our above work we made shape 2 out of only the inward bulge and the line segment AB, so for this task shape 2 would be ideal.

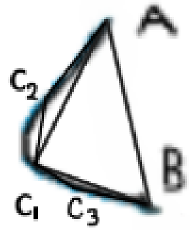
In this shape we can always find a point not on the line AB (If we had all points on the line AB then it would be on the boundary of the shape, which would mean that we would not be able to show the shapes concavity with this line)

So, we are able to find a point not on the line AB but still on shape 2, let's call it C_1.

Now we have 3 distinct points A, B and C_1 so we can form a triangle ΔABC_1

In a triangle the sum of the length of two sides must be greater than the length of the third length i.e. $C_1A + C_1B > AB$.

Now the shape is divided into segments form C_1 to A and C_1 to B, on those intervals of shape 2 the points can either all be on their respective lines (C_1A or C_1B) or outside the line (either all points will be outside or only some). If it is outside, then we can form another triangle further dividing it into smaller intervals whose sum of lengths will be larger than the previous line and if yet we don't get all points on the line then we repeat the process forming smaller and smaller segments.



So, $C_1C_2 + C_2A > C_1A$ and $C_1C_3 + C_3B > C_1B$ and if this is true then

$C_1C_2 + C_2A + C_1C_3 + C_3B > C_1A + C_1B > AB$ and a similar thing happens as we subdivide further, this would mean that the sum of all subdivisions regardless how many subdivisions will always be greater than the length of the line segment AB.

Each time we make a triangle we would be increasing the combined length of all line segments joining the intervals. The smaller we make our interval the closer it gets to the actual length of that segment in that interval. Due to this the sum of all lines joining the interval becomes a better and better approximation of the length of the curve.

The perimeter of shape 1 and the original shape just differs by the part between A and B, as we have already showed that the part between A and B has higher length on the original shape than the on the chord AB present in shape 1, from this we can conclude that the perimeter of shape 1 is smaller then the perimeter of the original shape.

Comparing the areas of Shape 1 and the Original Shape

We have shown that shape 1 has lower perimeter than the original shape but what about the area?

As shape 1 is made by combining the area of both shape 2 and the original shape it has the area of both their areas combines so the overall area is larger than the original shape.

Using the above work to show that the shape which maximises area for a given perimeter can't be a Concave Shape

For all of the above work to be used for showing that the shape which maximises area for a given perimeter cannot be concave, we would need the perimeter to be the same and the area to remain higher than the original shape.

For that we can just increase the size of the shape 1, when we change the size of shape 1 to have the same perimeter as our original shape, the area is also increased. This is because our original shape had a higher perimeter so we must increase the size of shape 1 to match it. When we increase the size the of shape 1 it's which was already larger gets scaled up even more. So for every Concave shape we can find another shape which has the same perimeter with larger area.