The Sourdough Manifold

By: Chin Shun Yik

1 Introduction

Dear readers, I should start with a warning: I hope you've had a substantial meal before reading this. If not, this essay will definitely make you hungry. But as we are about to discover, the allure of a golden crusty loaf isn't just in the aroma — it's in the topology.

Most people see a loaf of sourdough and think of wild yeast, fermented flour and the perfect smear of salted butter. I see a high-pressure manifold under extreme thermal stress. To the untrained eye, the 'oven spring' — that dramatic expansion of dough in the first ten minutes of baking is a culinary miracle. But to a mathematician, it is a complex problem of differential geometry.

Bready or not, here we go! As it turns out, the secret to the perfect crumb isn't just in the starter — it's in the calculus.

2 Thermodynamics: The Engine of Expansion

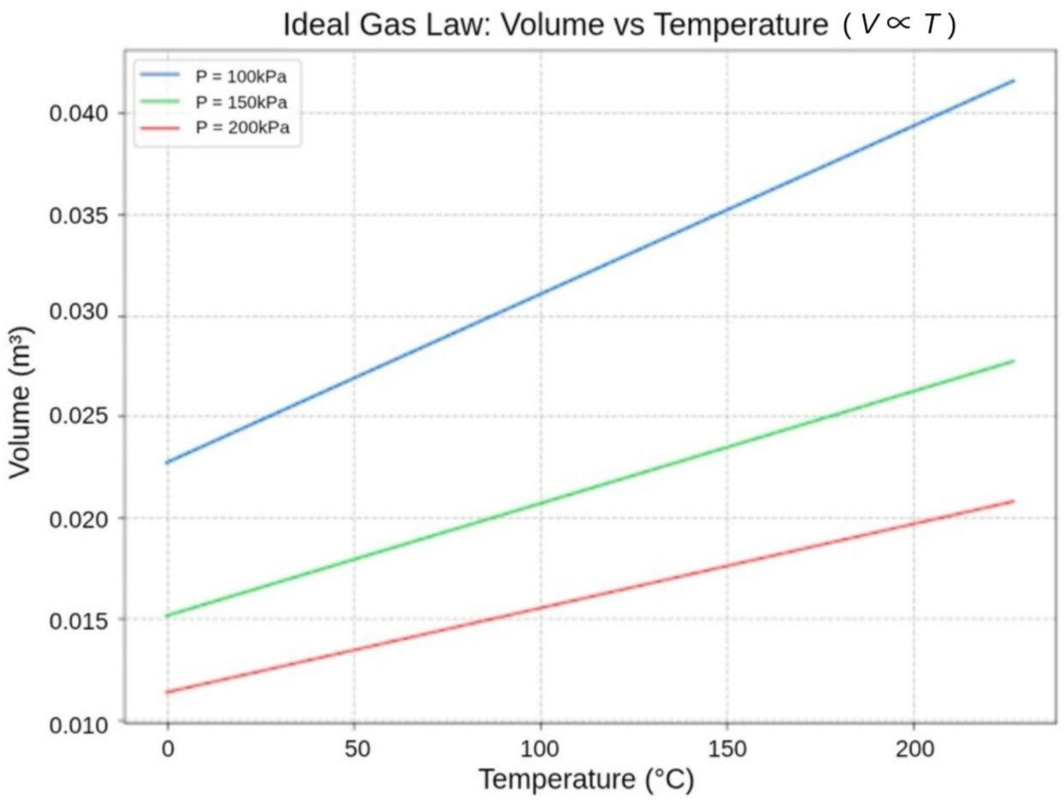
Before we can discuss the shape of the bread, we must address the engine of its expansion. As the dough enters the pre-heated oven, the temperature (T) rises from 25°C to over 200°C almost instantly. We can model the expansion of CO_2 and steam using the Ideal Gas Law:

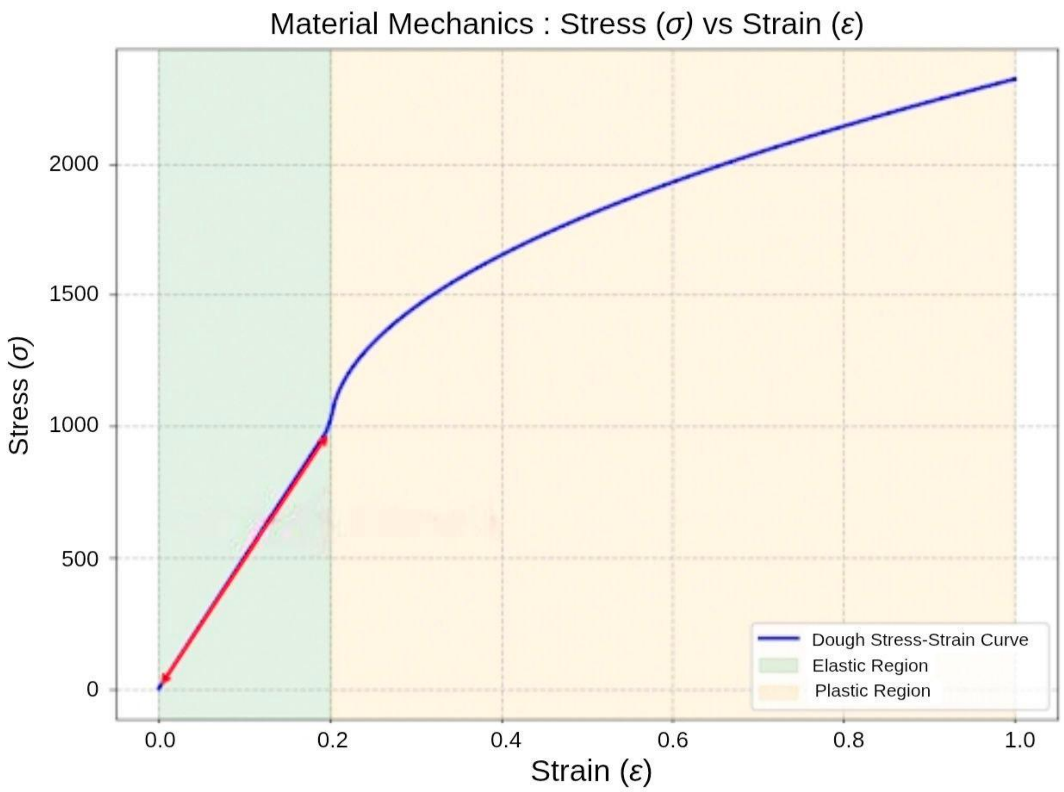
$$PV = nRT$$

Now, I don't want to get "yeast" bit over-excited, but as T increases, V also increases. However, dough is not an ideal container; it is a viscoelastic medium. The rate of volume change $\frac{dV}{dt}$ is constrained by the Young's Modulus (E) of the gluten network. The stress (σ) exerted by the gas must overcome the elastic resistance:

$$\sigma = E \cdot \ln\left(\frac{L}{L_0}\right)$$

where L/L_0 is the extension ratio. If E is too high due to low hydration, the work done by the gas $(W = \int P\, dV)$ is insufficient to cause significant expansion, resulting in a dense result. The baker's goal is to maximize the integral of this work before the proteins denature and the structure sets.


Ideal Gas Law: Volume vs Temperature ($V \propto T$)
P = 100kPa
P = 150kPa
P = 200kPa
0.040
0.035
0.030
0.025
0.020
0.015
0.010
Volume (m³)
0
50
100
150
200
Temperature (°C)


Material Mechanics : Stress (σ) vs Strain (ε)
2000
1500
1000
500
0
Stress (σ)
Dough Stress-Strain Curve
Elastic Region
Plastic Region
0.0
0.2
0.4
0.6
0.8
1.0
Strain (ε)

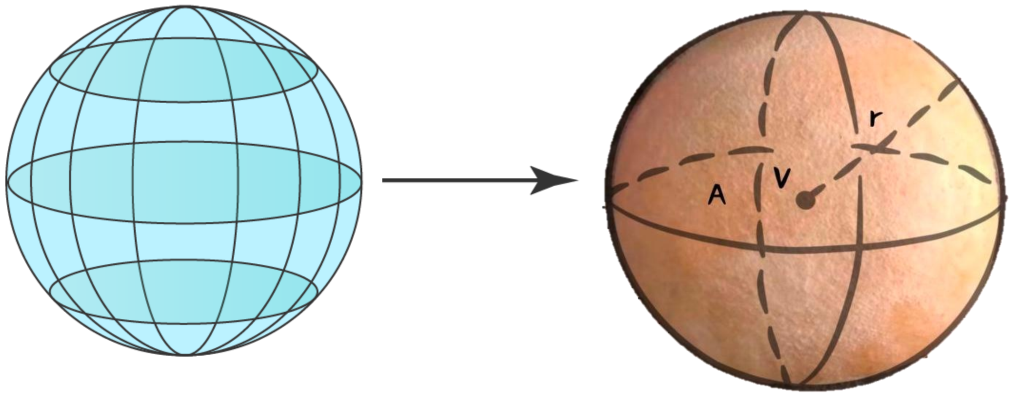
3 The Curvature Constraint: Why Bread "Explodes"

To understand why a baker must "score" their dough, we have to look at Gauss's Theorema Egregium. This theorem states that the Gaussian curvature (K) of a surface depends solely on intrinsic measurements within the surface. This means that it doesn't change no matter how you bend or fold it, as long as you don't stretch or tear it.

A raw boule of sourdough is topologically a sphere. As the gases push outward, the surface area (A) must increase to accommodate the new volume (V). According to the Isoperimetric Inequality in three dimensions:

$$36\pi V^2 \leq A^3$$

The sphere is the most efficient shape for containing volume, but a rigid and drying crust cannot increase its surface area fast enough to satisfy the inequality as V grows during the oven spring. Hence, the pressure will find the weakest point in the gluten network and rupture it randomly. This is the "topological crisis" of the sourdough: it must change its shape or break.



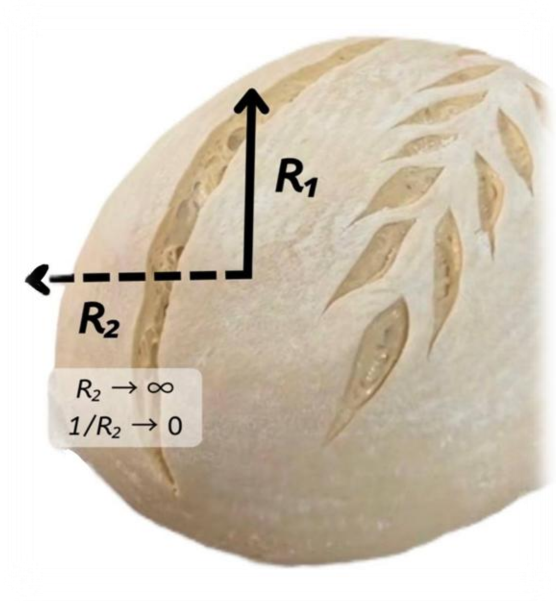
4 Scoring and the Young-Laplace Equilibrium

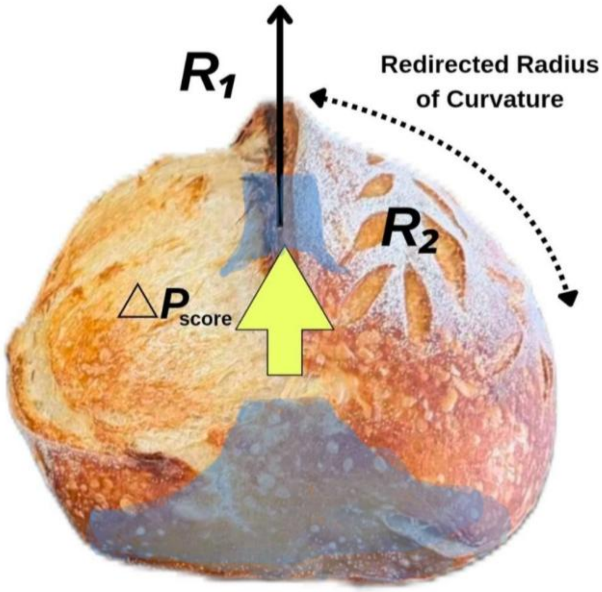
In the Young-Laplace equation, the pressure difference (ΔP) is inversely proportional to the principal radii of curvature (R_1, R_2)

$$\Delta P = \gamma\left(\frac{1}{R_{1},} + \frac{1}{R_2}\right)$$

where:

- γ is the surface tension of the dough
- R_1 and R_2 are the principal radii of curvature

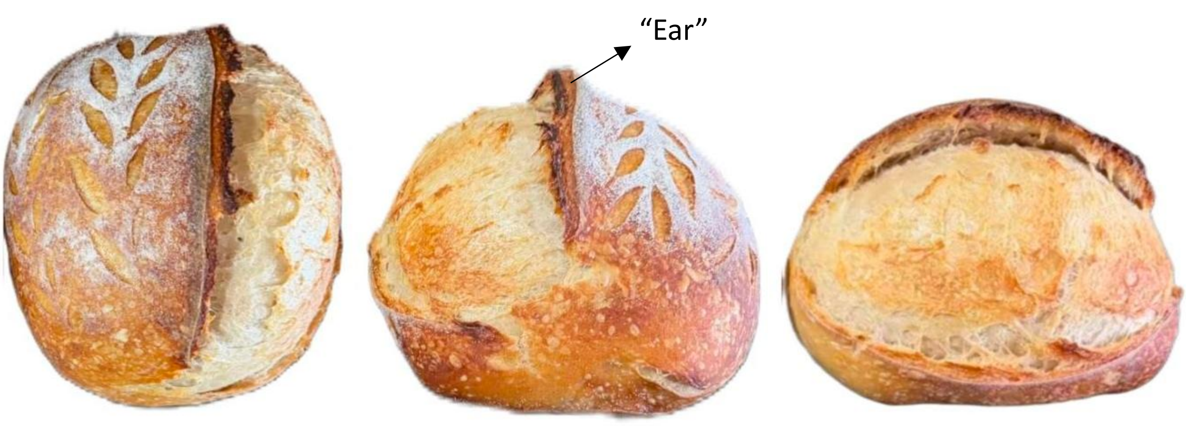




A flat cut creates a region where the radius (R) is effectively infinite along the length of the slice. This creates a path of least resistance. As the dough expands, it doesn't tear randomly. Instead, it unrolls along the score. This transforms the dough from a simple sphere into a more complex shape — often a prolate spheroid with a prominent "ear."

In a perfectly smooth boule, the radii are small and uniform which keeps the internal pressure high. By scoring the dough, the baker creates a localized region where the radius (R) along the cut effectively approaches infinity ($R \to \infty$).

Mathematically, this reduces the local pressure required for the dough to expand. The "ear" of the sourdough is a fascinating geometric artifact: a thin flap of dough forced upward and outward, creating a high surface area ridge that caramelizes via the Maillard reaction. By manipulating R, the baker directs the flow of energy.



Top view Front view Side view

5 Stochastic Modelling of the Crumb

If the crust is a problem of differential geometry, the interior crumb is a masterpiece of stochastic tiling. When you slice into a loaf, you are looking at a Voronoi Tessellation in three dimensions.

We define a set of nucleation point $S = \{p^1, p^2, \dots, p_n\}\ in\ \mathbb{R}^3.\ Each\ air\ cell\ R_k$ is defined as

$$R_k = \left\{x \in \mathbb{R}^3 \,\middle|\, \|x - p_k\| \leq \left\|x - p_j\right\|\ for\ all\ j \neq k\right\}$$

Don't "loaf" around on this part! The distribution of these cell sizes follows a Power Law rather than a Gaussian curve in a sourdough. If $N(d)$ is the number of pores of diameter d:

$$N(d) \propto d^{-D}$$

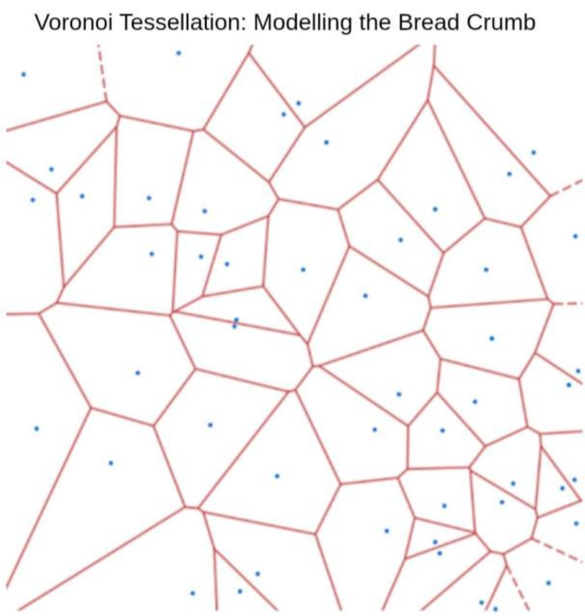
Where D is the Hausdorff Dimension. A higher D (typically 2.5–2.8) indicates a more complex and fractal-like internal surface area.

As the gas bubbles grow, they begin to press against their neighbours. The boundary where two bubbles meet is a flat plane, equidistant from their centres. We can quantify the "airiness" of the sourdough using the Euler-Poincaré Characteristic (χ):

$$\chi = V - E + F$$

where:

- V is the number of vertices
- E is the number of edges
- F is the number of faces in a polyhedral representation of the space



In an open-crumb sourdough, the connectivity of the holes is so high that χ becomes a negative value. We can also model the growth of these bubbles using the Avrami Equation which describes phase transformation:

$$1 - Y = e^{-kt^n}$$

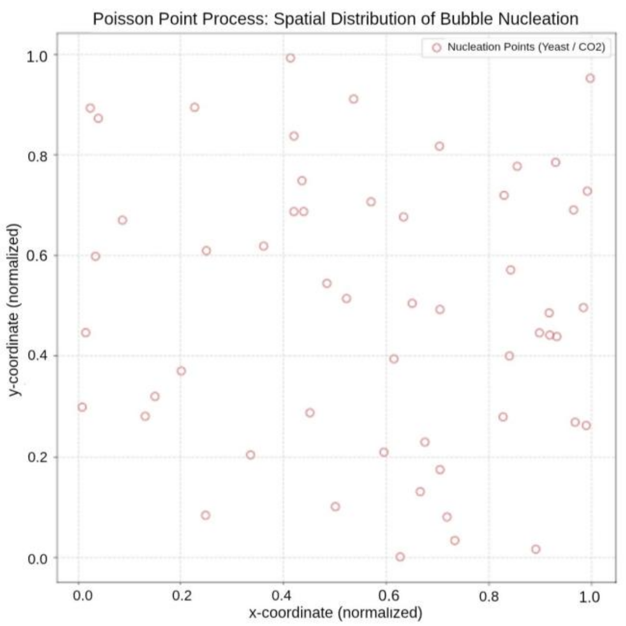
where Y is the fraction of the dough that has transformed into a solidified cellular foam. This equation proves that the perfect crumb is a function of time (t) and a growth constant (k) that depends on the baker's precise temperature control.

6 Probability Density and Hydration

While the Voronoi tessellation gives us the "skeleton" of the bread, it does not account for the sheer unpredictability of yeast. As a biological leavening agent, yeast does not produce gas in a uniform grid. Instead, it follows a Poisson Point Process. If we define λ as the average number of CO_2 nucleation points per cubic centimeter of dough, the probability of finding exactly k bubbles in a given volume V is given by:

$$P(X = k) = \frac{(\lambda V)^k e^{-\lambda V}}{k!}$$

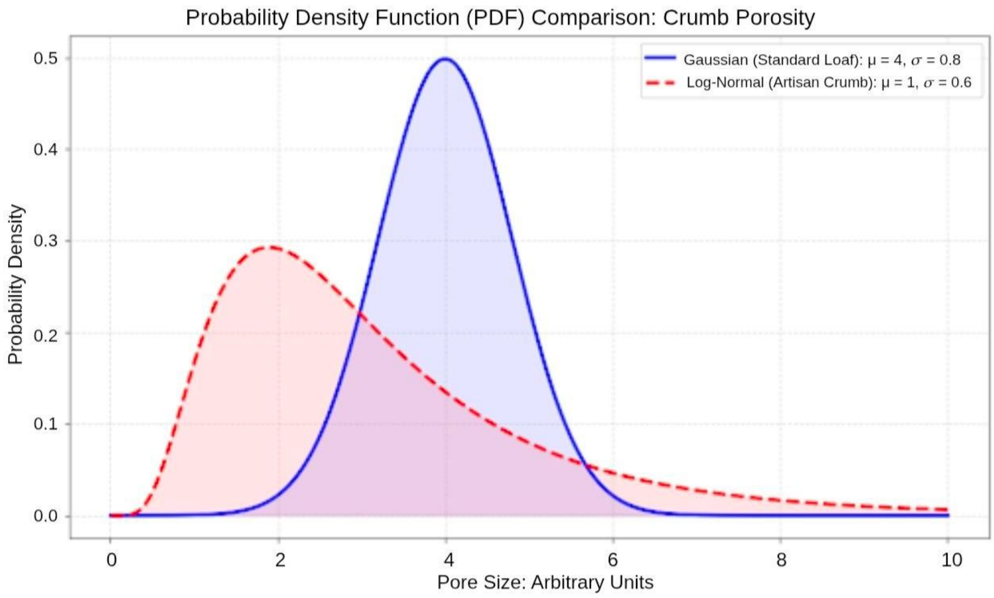
This randomness is what gives sourdough its "artisanal" look. However, the size of these bubbles is heavily dictated by the Hydration Ratio (H). In mathematical terms, hydration acts as a scaling factor for the dough's viscosity. A high-hydration dough (where $H > 0.75$) has a lower dynamic viscosity (μ) allowing bubbles to merge through a process called coalescence.



We can model the distribution of hole sizes using a Probability Density Function (PDF). In a dense sandwich loaf, the PDF resembles a narrow Gaussian Distribution (a Bell Curve), where the mean radius μ is small and the variance σ^2 is low.

In contrast, a "wild" sourdough follows a Log-Normal Distribution. This explains why you have a vast number of tiny micro-bubbles and a few "hero" bubbles that take up a disproportionate amount of space.

The transition from a Gaussian crumb to a Log-Normal crumb is a signature of non-Newtonian fluid dynamics at play during the mixing stage. By adjusting the hydration, the baker is essentially tuning the "variance" of a statistical field.

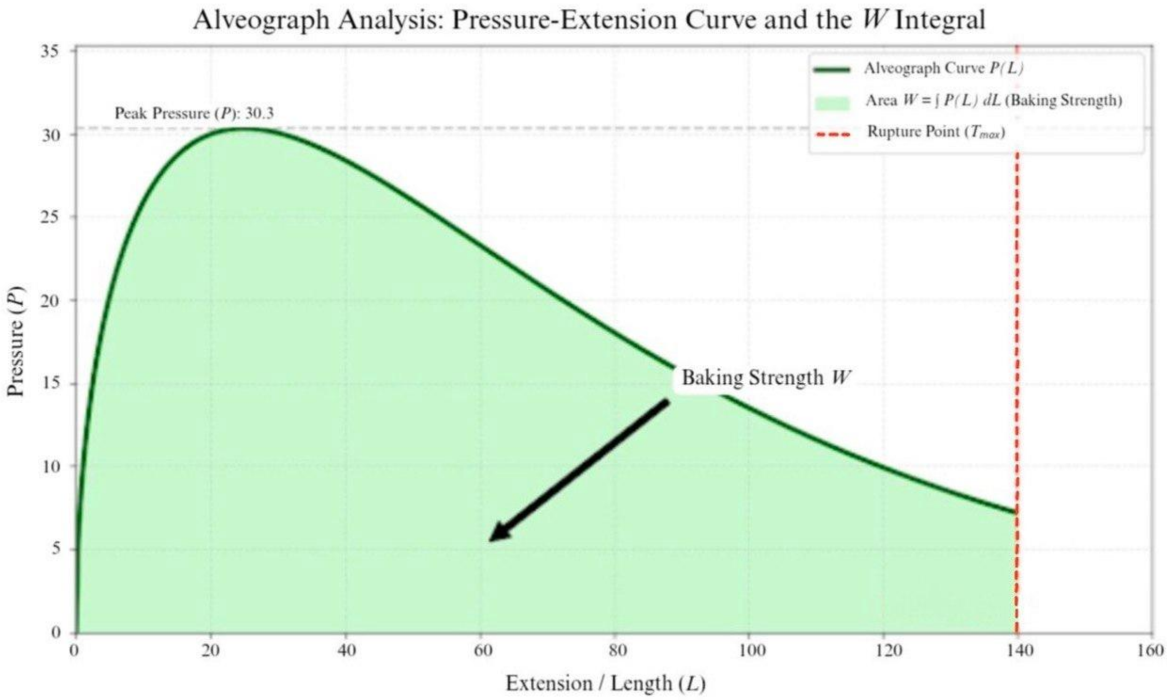


7 The Villain: The Collapse of the Gluten Manifold

Every mathematician needs a counter-example, and in baking, that is the "collapsed" loaf. This is a failure of linear elasticity. We can model the gluten strands as a network of interconnected tensors. Under normal conditions, the dough obeys Hooke's Law ($F = kx$), where the restoring force of the gluten balances the internal pressure.

However, if the dough is over-proofed, it enters the Plastic Deformation regime. At this point, the stress-strain curve flattens, and the material no longer returns to its original shape. If we plot the "Tenacity" (P) against the "Extensibility" (L) of the dough—a common test in professional milling known as the Alveograph — we can calculate the "Baking Quality" (W) as the area under the curve:

$$W = \int_0^{L_{max}} P(L)\, dL$$



If W is too low, the gluten manifold lacks the energy density to support its own weight. Topologically, the thousands of individual Voronoi cells merge into one large and flat void via Ostwald Ripening — a phenomenon where large bubbles "eat" smaller ones to minimize surface energy. The result is a dense and mathematically uninteresting pancake which is considered a tragic violation of geometric potential.

8 A Case Study in Topological Failure: The “Frisbee” Effect

To truly appreciate the elegance of a mathematically perfect loaf, one must first experience the "Frisbee" — the nickname bakers give to a loaf that fails to rise and spreads into a flat and dense disc. In my own kitchen, I once produced a Frisbee that served to diagnose structural instability. Upon slicing into this failed manifold, the internal geometry was revealing. Instead of a Log-Normal distribution of pores, the crumb was a "closed" system of tiny and uniform bubbles.

Using the Deborah Number (*De*), which relates the relaxation time of a material to the timescale of the observation, I realized my mistake:

$$De = \frac{t_r}{t_o}$$

My observation time (the bake) (t_o) was too short, meaning the gluten's relaxation time (t_r) dominated. Hence, I can conclude that:

- If $De \gg 1$, the dough behaves like a solid and cracks.
- If $De \ll 1$, the dough behaves like a fluid and puddles.
- If $De \approx 1$, the dough is fluid enough to expand but solid enough to hold its shape — a dynamic equilibrium.

Furthermore, my scoring was shallow. Instead of creating a singular line of infinite radius ($R \to \infty$), I had created a superficial scratch that quickly "healed" as the proteins coagulated. This forced the internal gases to exert pressure against the entire inner surface of the crust equally.

Since the crust had already begun its phase transition into a solid, the result was a stalemate: the pressure couldn't break out and the dough could only expand laterally across the baking stone. It was a vivid and edible demonstration of constrained optimization gone wrong. This failure taught me that baking isn't just following a recipe; it is the act of managing a complex system of simultaneous differential equations.

My first ever “frisbee”
(“knead”less to say it was a failure)

An example of a
sourdough when De ≫1

9 Conclusion: The Proof is in the Σ

We often separate "art" from "science," and "cooking" from "calculus." Yet, the sourdough loaf proves that these boundaries are as thin as a well-developed gluten window. From the Gaussian curvature of the crust to the fractal porosity of the crumb, every aspect of a loaf of bread is governed by the same laws that dictate the shape of the cosmos.

Next time you enjoy a slice, take a moment to appreciate the geometry. You aren't just eating bread; you are consuming a perfectly scored manifold. At the end of the day, no matter how you "slice" it, math makes everything better. Before we can bake, the dough must undergo a rigorous "proof" — in both the culinary and mathematical sense.

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