THE TRAJECTORY OF A SHUTTLECOCK

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1 Introduction

Badminton is an indoor Olympic racket sport where the aim is to hit a shuttlecock above a net into the opponent's half of the court. A shuttlecock is a light, cone-like object with a rounded, weighted base and a skirt of feathers. The feathers fan out, making the shuttlecock experience a high drag force, creating the illusion that the flight path of the shuttlecock is wildly unpredictable. It would seem that there is no way to mathematically map out the shuttlecock's trajectory, however, in this essay, I will show how the trajectory can be predicted using a range of formulas.

Figure 1: A feathered shuttlecock

Figure 2: A badminton court

2 Modelling a feathered shuttlecock as a projectile

Initially, I will model the shuttlecock as a "projectile", which can be defined as an object which is thrown and has the property that no other forces except the force of gravity acts on it. The object would therefore experience constant deceleration of g m/s^2 (where the letter g represents the gravitational field strength, with a numerical value of ~9.81).

If the initial speed of the trajectory is U, and the angle of projection to the horizontal is α, then the two equations for the motion are:

$$(1)\ x = U\cos(\alpha)\,t$$

$$(2)\ y = U\sin(\alpha)\,t - \frac{1}{2}gt^2$$

Where x and y are the horizontal and vertical components respectively.

Rearranging equation (1):

$$(3)\ t = \frac{x}{U\cos(\alpha)}$$

Substituting equation (3) into equation (1):

$$y = \frac{xt\sin(\alpha)}{t\cos(\alpha)} - \frac{1}{2}g\left(\frac{x}{U\cos(\alpha)}\right)^2$$

Which simplifies to:

$$y = x\tan(\alpha) - \frac{gx^2}{2U^2\cos^2(\alpha)}$$

The above equation represents the flight path of a projectile. However, as mentioned previously, a shuttlecock experiences significant drag, and cannot be modelled as a projectile, such as a tennis ball. I will therefore create a more inclusive model, which takes the drag force into account (any other forces can be assumed to be negligible).

3 Adding drag to the equation

As per the projectile equation above, I will separate the resultant force into horizontal and vertical components. In the horizontal direction, only the horizontal component of drag will be in action. In the vertical direction, both the vertical component of drag and the complete component of weight act on the shuttlecock.

Using the equations for weight and drag $W = mg$ and $D = \frac{1}{2}\rho C_d A v^2$ respectively, and the information that the initial speed of the trajectory is v:

$$\text{Horizontal: } F_x = -\left(\frac{1}{2}\rho C_d A v^2 \cdot \frac{v_x}{v}\right)$$

$$\text{Vertical: } F_y = -mg - \left(\frac{1}{2}\rho C_d A v^2 \cdot \frac{v_y}{v}\right)$$

The angle to the horizontal (θ) cannot be used when considering drag because θ is always changing. In the projectile equation, θ is constant because the only force acting is weight so the velocity components stay the same throughout the flight; in contrast, adding drag means that θ is never constant, so velocity components must be described as v_x and v_y, which are the horizontal and vertical components of the velocity at a given time. Since drag is a vector, its magnitude is $\frac{1}{2}\rho C_d A v^2$ and the direction is $\frac{v_x}{v}$ or $\frac{v_y}{v}$.

Therefore, instead of writing $\frac{1}{2}\rho C_d A v^2 \cos(\theta)$ and $\frac{1}{2}\rho C_d A v^2 \sin(\theta)$, they can be more accurately written as $\frac{1}{2}\rho C_d A v^2 \cdot \frac{v_x}{v}$ and $\frac{1}{2}\rho C_d A v^2 \cdot \frac{v_y}{v}$, which simplify to:

$$\text{Horizontal: } F_x = -\left(\frac{1}{2}\rho C_d A v v_x\right)$$

$$\text{Vertical: } F_y = -mg - \left(\frac{1}{2}\rho C_d A v v_y\right)$$

The following constant values can be used to substitute into the above equations:

- Mass of shuttlecock (m) = **0.005kg**
- Density of air (ρ) = **1.225kg/m^3** (assuming standard sea level density)
- Drag coefficient of shuttlecock (C_d) = **0.6** (average values are 0.55-0.65. I will therefore use 0.6)
- Cross-sectional area of shuttlecock (A) = $\pi\left(\frac{d}{2}\right)^2$ = **0.00342m^2** (where skirt diameter (d) = 0.066m)

As the term $\frac{1}{2}\rho C_d A$ is in both equations, and is a constant value, I will write it in calculations as 'k' for simplicity, and substitute the numerical values in at the end.

4 Deriving the differential equations

As $F = ma$, where a is the temporal derivative of velocity $\left(\frac{dv}{dt}\right)$, the above equations can be rewritten as the following differential equations to show the forces acting on a shuttlecock:

$$\text{Horizontal: } m\frac{dv_x}{dt} = -kvv_x$$

$$\text{Vertical: } m\frac{dv_y}{dt} = -mg - kvv_y$$

The next step is to simplify and then integrate it in order to solve it.

From the modified resultant force equations above, dividing through by m means that they are in their simplest forms, resulting in the following four differential equations:

$$(1)\ \frac{dv_x}{dt} = -\frac{k}{m} v v_x$$

$$(2)\ \frac{dv_y}{dt} = -g - \frac{k}{m} v v_y$$

Velocity (v) is the temporal derivative of position, and can be written as:

$$(3)\ \frac{dx}{dt} = v_x$$

$$(4)\ \frac{dy}{dt} = v_y$$

Because v is dependent on v_x and v_y, equations 1) and 2) involve both v_x and v_y. This means that both differential equations depend on each other, creating what is known as a "coupled differential equation". This is incredibly significant, as it means that they cannot be integrated and substituted into one another, and therefore there is no simple algebraic solution. Ultimately, the only way around this is to formulate a near-accurate numerical approximation. This is where the **Forward Euler Method** comes in.

5 The Forward Euler Method

In principle, this is a numerical approximation approach that uses repeated tangent lines to estimate the y-value for any given x-value. If the derivative of a curve and the y-value for a corresponding x-value are given, then the formula is:

$$y_{n+1} = y_n + hf'(x_n)$$

The four differential equations can then be rewritten in terms of the Forward Euler Method:

$$(1)\ v_{x,n+1} = v_{x,n} - \frac{k}{m} v v_{x,n} \Delta t$$

$$(2)\ v_{y,n+1} = v_{y,n} - \left(g + \frac{k}{m} v v_{y,n}\right) \Delta t$$

$$(3)\ x_{n+1} = x_n + v_{x,n} \Delta t$$

$$(4)\ y_{n+1} = y_n + v_{y,n} \Delta t$$

These are the formulae that I will be using to create the model. For a certain value of x (and the corresponding value of y), going up in small increments of x (namely increments of Δt), finding the y-values at those points, and then plotting a graph connecting all the data points, will display the trajectory of the shuttlecock in graphical form. Whilst this is simply an approximation, it can be adjusted so that the value of Δt is as small as possible, as smaller changes will lead to a more accurate result. If the change is small enough, then the method turns from numerical approximation into near-accurate perfection. Hence, I have selected Δt to be 0.001s.

6 Formation of Model

To formulate the proposed model, which will predict the trajectory of the shuttlecock, it is necessary to define the initial variables of position (x_0 and y_0) and velocity as the starting values for the Forward Euler Method. Since velocity is a vector, this can be written as the initial speed v_0 and the initial angle θ_0, which are calculated with the formulae:

$$v_0 = \sqrt{v_{x,0}^2 + v_{y,0}^2}$$

$$\theta_0 = \tan^{-1}\left(\frac{v_{y,0}}{v_{x,0}}\right)$$

Although velocity is split into its horizontal and vertical components in the differential equations ($v_{x,0}$ and $v_{y,0}$), displaying velocity as v_0 and θ_0 makes it more intuitive to understand, especially in later sections.

By displaying the model of the shuttlecock with drag versus without drag (the projectile model explained in Section 2), I can make visual comparisons between both trajectories. The models can be further compared by calculating the time of flight, total distance travelled, and maximum height that the shuttlecock/projectile reaches.

For the shuttlecock, this is done by using the formulae below:

- Time of flight = $\sum_{x=x_0}^{x_{final}} \Delta t$
- Total distance travelled = $x_{final} - x_0$
- Maximum height = y_{max}

In contrast, for the projectile, first it is necessary to derive the general equation for the flight path of the projectile:

$$y_{landing} - y_0 = (x_{landing} - x_0)\tan(\theta_0) - \frac{g(x_{landing} - x_0)^2}{2v_0^2\cos^2(\theta_0)}$$

The above quadratic equation is a translated version of the projectile trajectory formula from Section 2.

Next, by solving the equation for the roots p and q (where $p > q$), the following equations can be derived:

- Time of flight = $\frac{p-x_0}{v_{x,0}}$
- Total distance travelled = $p - x_0$
- Maximum height = $\begin{cases} \frac{v_{y,0}^2}{2g}, & x_0 < \frac{p+q}{2} \\ y_0, & x_0 \geq \frac{p+q}{2} \end{cases}$

For the maximum height, if x_0 is less than the midpoint of p and q, then the maximum is yet to come and therefore the maximum is the point where $v = 0$ i.e. $y = \frac{v_{y,0}^2}{2g}$. If x_0 is equal to or greater than the midpoint of p and q, the maximum is y_0, because all points from there are decreasing.

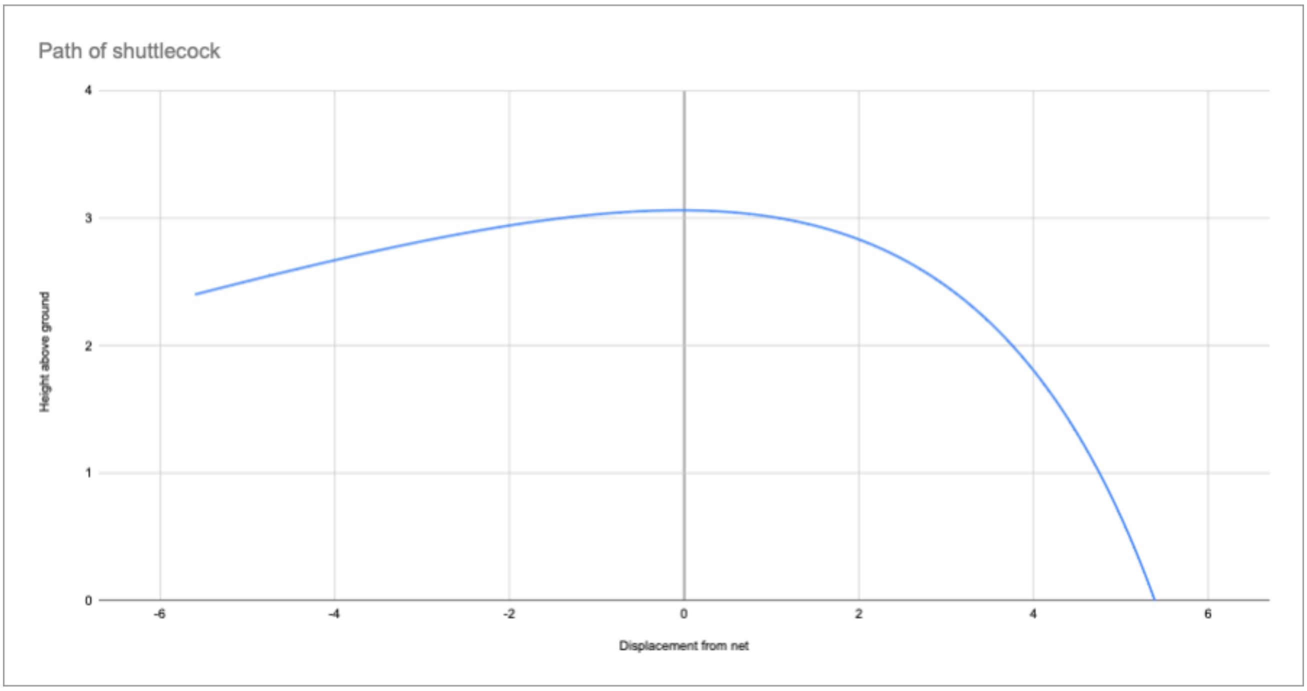

Figure 3: Graph showing trajectory of a shuttlecock

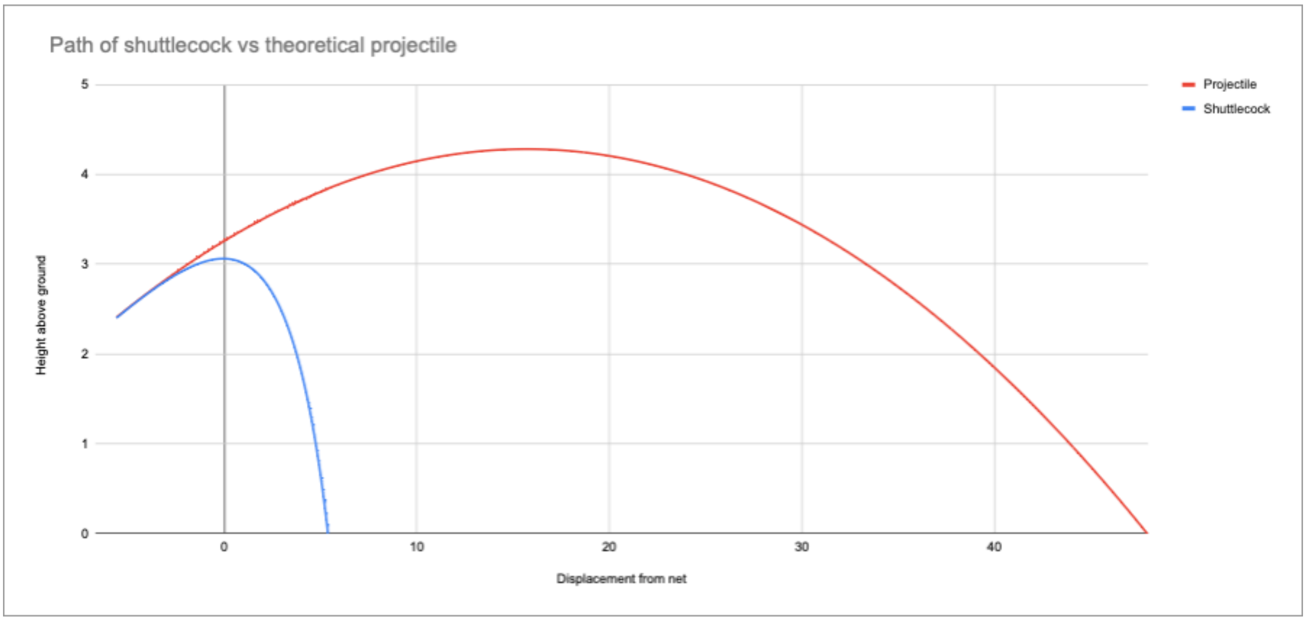

Figure 4: Graph showing trajectory of a shuttlecock versus a projectile

The graphs above illustrate the flight path with over a thousand data points, which have been obtained using the Forward Euler equations (Section 5) and represent the position at each millisecond of time after the start of the flight. Due to the small value of Δt, the points on the graph are extremely close and appear to merge into one smooth line, displaying an almost perfect visual representation of the flight of a shuttlecock. The graph shows that it is possible to map out the path of a shuttlecock mathematically, even when the values are numerical approximations.

7 Testing the model

I further tested the newly created model to determine if it can be used for any type of shot in badminton, and if it can be applied for different materials of shuttlecock. The shot types that I observed are:

- **Dropshot** - a slow shot aimed just over the net
- **Smash** - a fast and powerful shot aimed slightly downwards
- **Clear** - a powerful shot aimed high towards the other end of the court

I also tested shuttlecocks which are **synthetic** versus those that are **feathered**.

7.1 Testing types of shots

To validate this model, I obtained five samples of each shot type from a match between Loh Kean Yew and Viktor Axelsen (previously the world and Olympic champions respectively). I recorded the initial position at the time that the racket makes contact with the shuttlecock, the final position, and the change in position over a small number of frames to calculate the initial velocity.

I used the recordings to create a graph with each trajectory on it, where:

- Green is dropshot
- Red is smash
- Blue is clear

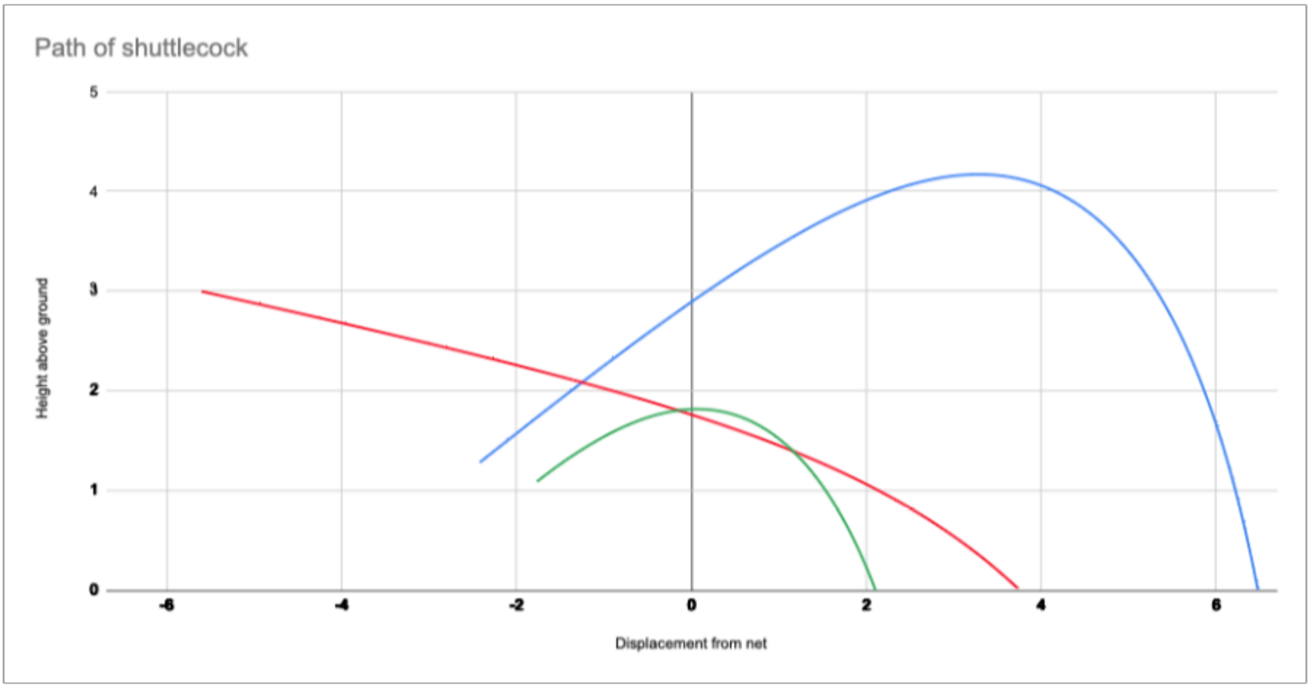

Figure 5: Graph showing different types of badminton shots

As expected, the graph shows that each shot follows a parabolic path, which proves that the model is an accurate representation of all types of shots. The following observations can be seen from the graph:

- The **clear** ends its trajectory very close to the edge of the court, because its purpose is to send the opponent all the way back to the end of the court to reach it.
- The **dropshot** hits the ground just after 2m from the net, which means it barely makes the service box.
- The **smash** appears to be almost like a straight line downwards, even though one may expect it to be a curve like the other two. This is because the shuttlecock travels so fast that the drag force can barely slow it down. As a result, it loses very little speed over the course of its flight.
- Both the clear and dropshot are aimed at very similar angles (34.91° versus 34.2° respectively), yet their paths seem very different when displayed visually. This is because the main factor differentiating the two is the speed of the shot: a dropshot is very slow (usually less than 10m/s), whereas a clear is faster (around 30m/s), hence why the clear travels higher and further, while the dropshot travels much lower.

7.2 Testing types of materials

Using the model, I tested the different shuttlecock materials by changing from feathered to synthetic; I also modified the drag coefficient from 0.6 to 0.55.

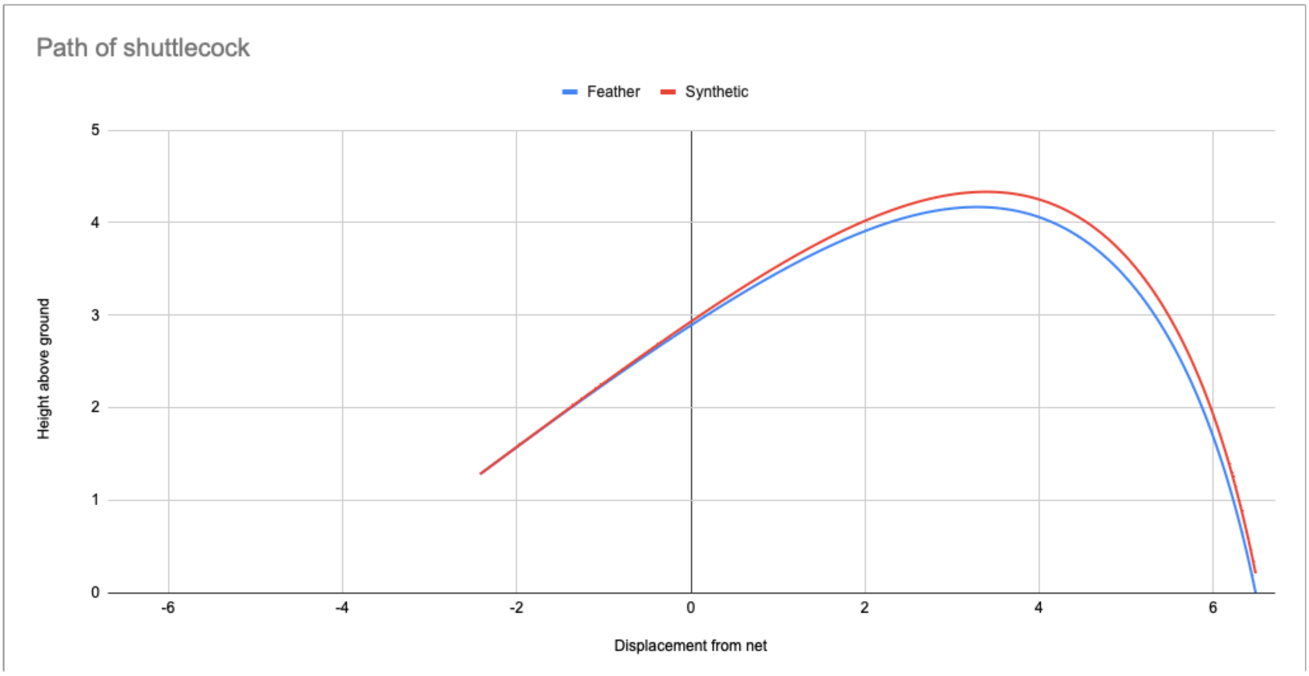

Figure 6: Graph showing the trajectory of a synthetic shuttlecock versus a feathered shuttlecock

The overall shapes of the flight paths are very similar, but the synthetic one travels slightly higher before beginning a quicker descent. The feather shuttlecock's trajectory is more defined, rendering it more suitable for professional play.

8 Conclusion

By using the Forward Euler method to approximate the differential equations, I have successfully created a model that accurately predicts the flight path of a shuttlecock, regardless of the type of shot taken, or the material of the shuttlecock being used. This model can be used for any object, where the only forces acting on it are weight and drag.

This essay has deepened my understanding of how badminton is not a random sport, and that even the most unpredictable concepts can be explained by mathematics.

9 References

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