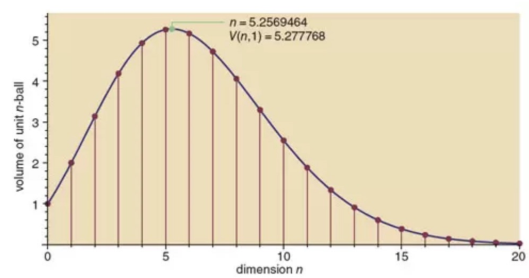
A 3D sphere has a large volume. $\frac{4}{3}\pi r^3$, everyone knows it - it's a football, it's a scoop of ice cream , it's the hot, glaring sun. However, what would happen if you increase the dimensions?

Take a sphere of radius 1:

Dimension	Volume
3	4.189
5	5.264
7	4.725
10	2.550
20	0.026



(graph from American Scientist)

A ball is meant to be substantial yet as dimensions increase, they effectively vanish. But intuitively, how is this possible? As dimensions increase, volume can spread out in more directions and the radius can't sustain its magnitude, resulting in the object thinning out to some degree. This strange paradox contains the mysterious workings of some calculus and its deep connections to the gamma function.

1 Let's start somewhere familiar.

The volume of a sphere seems to come from a random place. How does a fraction and the cube of pi equate to a volume?

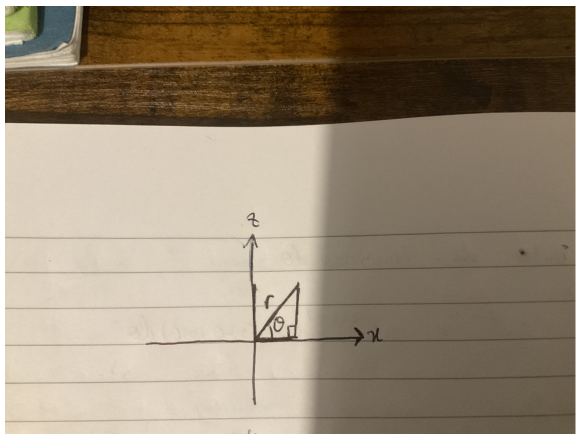
The equation is $x^2 + y^2 + z^2 = r^2$ and we can use this as a boundary for our integral.

$$\iiint_{x^2+y^2+z^2\leq R^2} 1\,dx\,dy\,dz$$

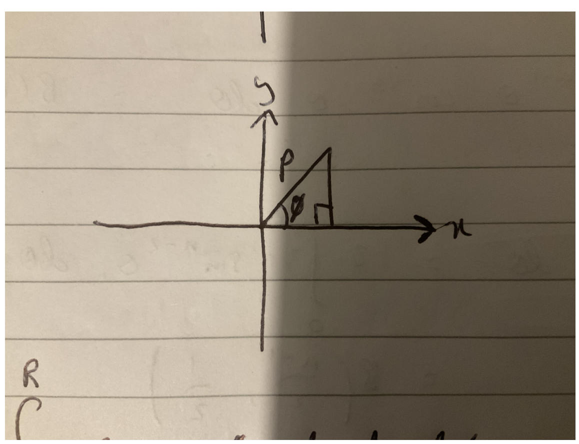
From this we can make another integral using spherical co-ordinates:

In the xyz plane, we have 2 ways we can rotate a shape, horizontally or vertically about the origin.

we can find a right angled triangle to link z and x.



Hence we can say that $z = r\cos\theta$ and the projection from the z axis is $r\sin\theta$. This projection p effectively becomes our new radius for a rotation vertically and we can now find x and y.



$x = p\cos\phi, y = p\sin\phi$

and therefore , we obtain

$x = r\sin\theta\cos\phi$ and $y = r\sin\theta\sin\phi$

This same method can be used to obtain coordinates of higher dimensions.

Each time we add a new axis the new $r\sin x$ becomes the radius we use to rotate through the other axis.

For example, in 4 D, we end up with

w $= r\cos\theta$,

z $= r\sin\theta\cos\phi$

$x = r\sin\theta\sin\phi\cos\omega$, $y = r\sin\theta\sin\phi\sin\omega$.

Since we changed co-ordinates we also need to find the volume scale factor (Jacobian).

This is done by finding the determinant of a tedious matrix.

$$J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial \phi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial \phi} \\ \frac{\partial z}{\partial r} & \frac{\partial z}{\partial \theta} & \frac{\partial z}{\partial \phi} \end{vmatrix} = \begin{vmatrix} \cos\theta\sin\phi & -r\sin\theta\sin\phi & r\cos\theta\cos\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\phi & 0 & -r\sin\phi \end{vmatrix}$$

(maths stack exchange, I switched θ and ϕ in my explanation)

This consists of a matrix of tangent vectors in which their det tells us the volume scale factor which is $= r^2\sin\theta$.

It's easy to see how this would get long and boring for higher dimensions.

Our integral now transforms to

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} r^2\sin\theta\, dr\, d\theta\, d\phi = \frac{4}{3}\pi R^3$$

2 Simplifying our Jacobian

Doing this for higher dimensions is a painful task so instead lets make it easier.

Before we start I need to clarify a few things. Our tangent vectors are perpendicular to each other. This is because when dotted together they equal 0 (suggests orthogonal)

For example, for a 3d sphere we have 3 tangent vectors , h_ϕ h_r h_θ.

$$h_n = \frac{\partial h}{\partial n}$$

$$h_\phi = (0,\ -r\sin\theta\sin\phi,\ r\sin\theta\cos\phi)$$

$$h_r = (\cos\theta,\ \sin\theta\cos\phi,\ \sin\theta\sin\phi)$$

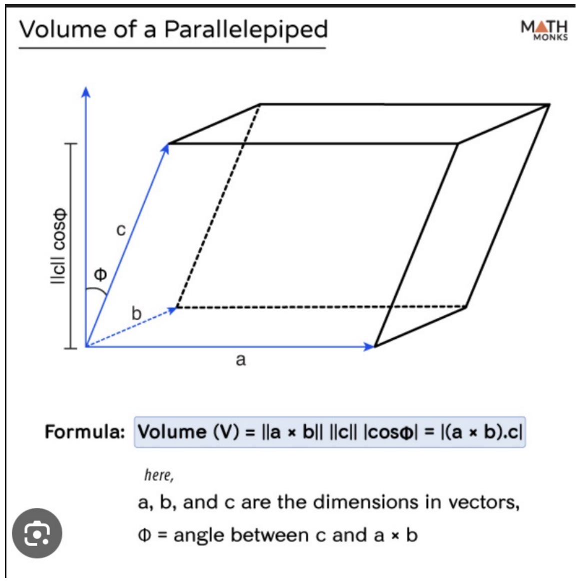
$$h_\theta = (-r\sin\theta,\ r\cos\theta\cos\phi,\ r\cos\theta\sin\phi)$$

$$h_\phi \cdot h_r = 0(\cos\theta) - r\sin^2\theta\sin\phi\cos\phi + r\sin^2\theta\cos\phi\sin\phi = 0$$

$$h_\phi \cdot h_\theta = 0(-r\sin\theta) - r\sin\theta\cos\theta\sin\phi\cos\phi + r\sin\theta\cos\theta\sin\phi\cos\phi = 0$$

$$h_r \cdot h_\theta = -r\sin\theta\cos\theta + r\sin\theta\cos\theta(\cos^2\phi + \sin^2\phi) = 0$$

As a result, our 3 tangent vectors form a parallelepiped and our determinant = volume of parallelepiped.



As such we can apply this to our vectors.
Let's define our unit vectors.

$$\hat{i} = \frac{h_r}{|h_r|} \qquad \hat{j} = \frac{h_\theta}{|h_\theta|} \qquad \hat{k} = \frac{h_\phi}{|h_\phi|}$$

$$\hat{i}|h_r| = h_r \qquad \hat{j}|h_\theta| = h_\theta \qquad \hat{k}|h_\phi| = h_\phi$$

Subbing into our formula we achieve:

$$\hat{k}|h_\phi| \cdot \left(\hat{i}|h_r| \times \hat{j}|h_\theta|\right) = |h_\phi||h_r||h_\theta|\,\hat{k} \cdot (\hat{i} \times \hat{j})$$

In this case we are simply left with the magnitude product as the volume of 3 unit vectors in a parallelepiped is just 1.

So for a 3D sphere we end up with

$$|h_r| = \sqrt{\cos^2\theta + \sin^2\theta\cos^2\phi + \sin^2\theta\sin^2\phi} = 1$$

$$|h_\theta| = \sqrt{r^2\sin^2\theta + r^2\cos^2\theta\cos^2\phi + r^2\cos^2\theta\sin^2\phi} = r$$

$$|h_\phi| = \sqrt{0 + r^2\sin^2\theta\sin^2\phi + r^2\sin^2\theta\cos^2\phi} = r\sin\theta$$

$$\to r^2 \sin\theta$$

Which is the same result as our jacobian.
In turn a Jacobian for n - dimensions would be

$$J_n = r^{n-1} \sin^{n-2}\theta_1 \sin^{n-3}\theta_2 \sin^{n-4}\theta_3 \cdots \quad dr\, d\theta_1\, d\theta_2\, d\theta_3 \cdots$$

Again just being the result of the product of the magnitudes.

3 the volume of a n - dimensional sphere

$$V = \int_0^R r^{n-1}\, dr \int_0^\pi \sin^{n-2}\theta_1\, d\theta_1 \int_0^\pi \sin^{n-3}\theta_2\, d\theta_2 \cdots \int_0^{2\pi} d\theta_{n-1}$$

$$= \frac{2R^n}{n} \int_0^{2\pi} d\theta_{n-1} \int_0^\pi \sin^{n-2}\theta_1\, d\theta_1 \int_0^\pi \sin^{n-3}\theta_2\, d\theta_2 \cdots$$

Now you might be wondering how we evaluate the endless sin integrals well…

4 This is where the magic happens

There's a function that Leonard Euler made that seems to solve our problems, the beta function.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

Our integrals look awfully similar to this so with a simple substitution we should find a solution.

$$\int_0^1 t^{x-1}(1-t)^{y-1}\,dt$$

$$t = \sin^2\theta \qquad dt = 2\sin\theta\cos\theta\,d\theta \qquad 0 \to 0 \qquad 1 \to \frac{\pi}{2}$$

$$2\int_0^{\pi/2} \sin^{2x-2}\theta\cos^{2y-2}\theta(\sin\theta\cos\theta)\,d\theta$$

$$= 2\int_0^{\pi/2} \sin^{2x-1}\theta\cos^{2y-1}\theta\,d\theta = B(x,y)$$

Notice that at y = $\frac{1}{2}$, the cos term vanishes and if we let 2x - 1 = n - a we find that x = (n - a + 1)/2. This is something that we can exploit.

For example,

$$\int_0^{\pi} \sin^{n-2}\theta_1\,d\theta_1 = 2\int_0^{\pi/2} \sin^{n-2}\theta_1\,d\theta_1 = B\left(\frac{n-1}{2},\frac{1}{2}\right)$$

What's also interesting is that the beta function and the gamma function are closely related.

$$B(x,y) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

5 What even is the gamma function?

The factorial is a basic concept. $n! = n(n-1)(n-2)(n-3)...$

But what's annoying is that n! can only take integers. You wouldn't be able to take n = 1/2 or any other fraction. The gamma function is just the factorial but extended to non integers.

Euler also discovered the Pi function.
$\Pi(x) = \int_0^1 (-\ln(t))^x dt$

This actually perfectly fit in with the factorial, at x = 1 it gave 1, at 2 it gave 2 and at 6 it gave 720. It was perfect. So how do we get to the gamma function?

If we let t = $\frac{1}{e^s}$ then $dt = \frac{-1}{e^s} ds$

As such our integral becomes
$\int_\infty^0 (-\ln(e^{-s})^x)(-(e^{-s}))ds$

$=\int_0^\infty s^x e^{-s} ds = \Gamma(x+1) = \Pi(x)$

$\therefore \int_0^\infty s^{x-1} e^{-s} ds = \Gamma(x)$

This means that for $(n-1)! = \Gamma(n)$ and this extends to non integers , one notable result is $\Gamma(\frac{1}{2}) = \sqrt{\pi}$

You might be asking how does a factorial apply to the volume of balls?
Well since $\Gamma(x)$ is just a continuous generalization of the factorial, it allows you to connect even and odd dimensions when accumulating volume .

6 Applying the Gamma function.

In the example above,

$$\therefore\ B\left(\frac{n-1}{2},\frac{1}{2}\right)=\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}$$

Now we can write out our entire function in terms of gamma functions

$$V=\frac{2R^n}{n}\frac{\Gamma\left(\frac{n-1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n}{2}\right)}\times\frac{\Gamma\left(\frac{n-2}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-1}{2}\right)}\times\frac{\Gamma\left(\frac{n-3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{n-2}{2}\right)}\cdots$$

In this moment of elegance, the expression telescopes. The Gamma functions cancel each other to leave us with:

$$V=\frac{2R^n}{n}\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}\right)}=\frac{R^n\pi^{n/2}}{\left(\frac{n}{2}\right)\Gamma\left(\frac{n}{2}\right)}$$

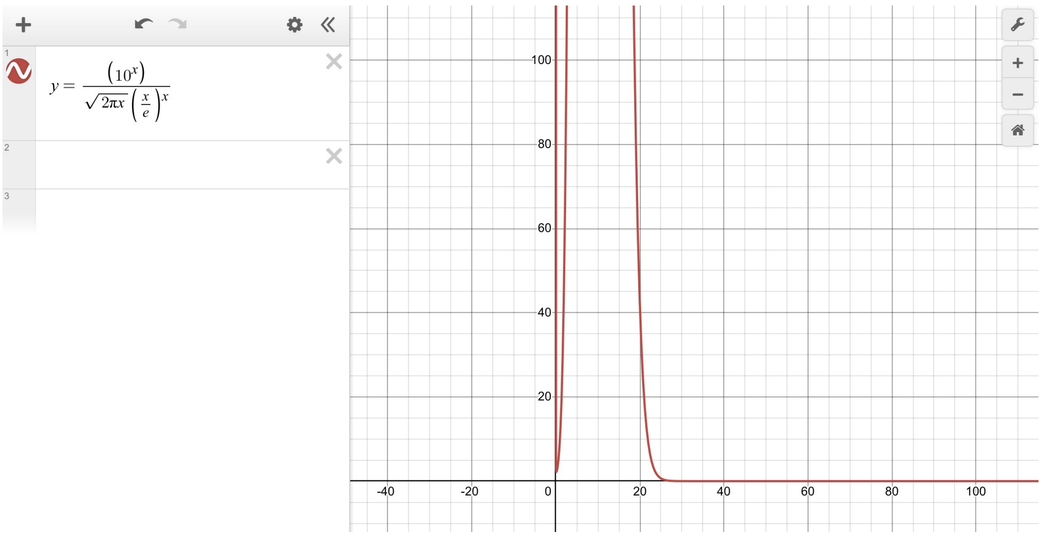
$$V=\frac{R^n\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$$

There's also a really nice property of the gamma function when n $\Gamma(n)=\Gamma(n+1)$ which highlights the beauty of this expression. From multi dimensional spheres to a simple fraction.

7 But why?

As n tends to infinity the gamma function dominates the numerator

(the denominator is called Stirling's formula for an approximation of the gamma function)



The peak is so large that it doesn't fit on my screen yet you can see how it immediately drops to the floor after 20.

As a result (using Stirling's formula)

$\Gamma(\frac{n}{2}+1)$ is $\approx \sqrt{2\pi(\frac{n+2}{2})}(\frac{n+2}{2e})^{\frac{n+2}{2}}$

which is $\approx \sqrt{\pi n}(\frac{n}{2e})^{\frac{n}{2}}$

as $n \to \infty$ the denominator grows faster than $R^n \pi^{\frac{n}{2}}$

So $V \to 0$

Conceptually, the sphere's radius is fixed but in higher dimensions the mass tends to move outward. As a result, the space it occupies becomes effectively negligible. Even weirder, as dimensions increase, the mass becomes extremely concentrated near the boundary or surface of the object. It means that the sphere becomes somewhat hollow. It's paradoxically beautiful yet beautifully paradoxical at the same time.