Why π Appears Where It Shouldn't

The Mathematics Behind the Basel Problem

Liam Somers

What begins as a question about square numbers turns into a story about circles, waves, prime numbers, and one of the most famous functions in mathematics.

Imagine adding the reciprocals of every square number:

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \cdots .$$

At first glance the problem looks purely arithmetic. Nothing here suggests circles, geometry, or trigonometry. The series appears to be nothing more than a collection of fractions built from square numbers.

Yet when mathematicians finally solved the problem, they discovered something extraordinary. The answer contains one of the most famous constants in mathematics:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This result is startling. The left-hand side contains only integers, while the right-hand side contains π, the constant most people associate with circles. Why should a sum involving square numbers have anything to do with geometry?

Even the first few terms hint that something surprising is happening. If we calculate partial sums, we obtain

$$1 = 1.0000, \qquad 1 + \frac{1}{4} = 1.2500, \qquad 1 + \frac{1}{4} + \frac{1}{9} \approx 1.3611,$$

and

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} \approx 1.4236.$$

After eight terms, the sum is already

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \frac{1}{36} + \frac{1}{49} + \frac{1}{64} \approx 1.5274.$$

The true value,

$$\frac{\pi^2}{6} \approx 1.644934,$$

is not far away. Even a small number of terms gives a surprisingly good estimate.

This identity is known as the **Basel problem**, one of the most beautiful results in mathematics. It begins with a simple infinite series but ultimately reveals deep connections between number theory, trigonometric functions, and analysis.

1. The Historical Challenge

The Basel problem was first posed in 1644 by the Italian mathematician Pietro Mengoli. At the time, mathematicians were only beginning to explore infinite series, and many questions about convergence were still poorly understood. The problem attracted the attention of several leading mathematicians, including members of the Bernoulli family, but despite considerable effort they were unable to determine the exact value of the series [2].

The breakthrough came in the 1730s when Leonhard Euler solved the problem at the age of twenty-eight. Euler's solution astonished the mathematical community because the appearance of π in the answer was completely unexpected. His reasoning was also unusually bold. He treated trigonometric functions as if they behaved like polynomials, manipulating infinite products in ways that had not yet been rigorously justified. Later mathematicians developed the formal foundations that confirmed Euler's intuition [1, 3].

Today the Basel problem is celebrated as one of the earliest examples of the hidden unity of mathematics: a result showing that ideas which seem unrelated may in fact be deeply connected.

2. Why the Series Converges

Before calculating the value of the series, it is important to understand why it converges at all.

Compare the Basel series

$$\sum_{n=1}^{\infty} \frac{1}{n^2}$$

with the harmonic series

$$\sum_{n=1}^{\infty} \frac{1}{n}.$$

Although both series have terms that approach zero, the harmonic series diverges. The difference lies in how quickly the terms decrease.

More generally, mathematicians study the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p}.$$

This series converges when $p > 1$ and diverges when $p \leq 1$. Since the Basel series has $p = 2$, it lies safely on the convergent side of the boundary [1].

This already reveals something interesting: a tiny change in the exponent dramatically changes the behaviour of the series. Replacing $1/n$ with $1/n^2$ is enough to turn divergence into convergence.

3. Why the Series Converges So Quickly

Another way to understand the behaviour of the series is through integrals. Consider

$$\int_n^{\infty} \frac{1}{x^2}\, dx.$$

Evaluating the integral gives

$$\int_n^{\infty} \frac{1}{x^2}\, dx = \left[-\frac{1}{x}\right]_n^{\infty} = \frac{1}{n}.$$

This tells us that the remainder of the series after n terms behaves roughly like $1/n$. So after 10 terms the error is around 0.1, and after 100 terms it is around 0.01. This explains why the partial sums approach the limit relatively quickly.

That does not tell us the exact value of the series, but it explains why numerical approximations become accurate so fast. The series is not just convergent; it is efficiently convergent.

4. Euler's Brilliant Insight

Euler's key idea was to examine the function

$$\frac{\sin x}{x}.$$

The sine function has zeros at

$$x = \pm\pi, \pm 2\pi, \pm 3\pi, \ldots$$

and this pattern resembles the roots of a polynomial. If an ordinary polynomial has roots r_1, r_2, r_3, it can be written as

$$(x - r_1)(x - r_2)(x - r_3).$$

Euler reasoned that $\sin x$ might behave like a polynomial with infinitely many roots. This led him to propose the remarkable identity

$$\frac{\sin x}{x} = \prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

At the time, this was daring mathematics. Euler did not have a modern rigorous proof for manipulating such an infinite product. Yet his instinct was correct, and modern analysis confirms the identity [3, 4].

Now compare this with the Maclaurin expansion of $\sin x$:

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots.$$

Dividing by x gives

$$\frac{\sin x}{x} = 1 - \frac{x^2}{6} + \frac{x^4}{120} - \cdots.$$

The coefficient of x^2 here is $-1/6$.

Next, look at the product

$$\prod_{n=1}^{\infty} \left(1 - \frac{x^2}{n^2\pi^2}\right).$$

If we expand it, the coefficient of x^2 comes from choosing the term $-x^2/(n^2\pi^2)$ from one factor and 1 from all the others. So the coefficient of x^2 is

$$-\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2}.$$

Since both expressions represent the same function, the coefficients must be equal:

$$-\sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} = -\frac{1}{6}.$$

Multiplying by $-\pi^2$ gives the result

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is one of the most elegant moments in mathematics. A problem involving square numbers suddenly reveals a hidden connection with trigonometric functions and the geometry of circles.

5. A Second Perspective: Fourier Series

One of the reasons the Basel problem is so beautiful is that it can be approached in more than one way. Another route comes from **Fourier analysis**, the study of how functions can be decomposed into sums of sine and cosine waves.

Consider the function

$$f(x) = x^2$$

on the interval $[-\pi, \pi]$. Its Fourier series expansion is

$$x^2 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos(nx).$$

If we substitute $x = 0$, then $\cos(0) = 1$, so we obtain

$$0 = \frac{\pi^2}{3} + 4 \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

From this, one can rearrange the alternating sum and recover

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}.$$

This is striking because the same result emerges from a completely different branch of mathematics. Euler's argument uses roots of $\sin x$, while the Fourier argument uses wave-like expansions of functions. The Basel problem therefore sits at a crossroads: it is not merely an isolated trick, but a meeting point of multiple mathematical ideas.

6. A Geometric Interpretation via Parseval's Identity

The Basel problem also appears naturally through **Parseval's identity**, which relates the "energy" of a function to the squares of its Fourier coefficients.

For the function $f(x) = x$ on $[-\pi, \pi]$, Parseval's identity gives

$$\frac{1}{\pi} \int_{-\pi}^{\pi} x^2 \, dx = 2 \sum_{n=1}^{\infty} \frac{1}{n^2}.$$

Now compute the integral:

$$\frac{1}{\pi}\int_{-\pi}^{\pi} x^2\,dx = \frac{1}{\pi}\cdot 2\int_{0}^{\pi} x^2\,dx = \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2}{\pi}\cdot\frac{\pi^3}{3} = \frac{2\pi^2}{3}.$$

So

$$\frac{2\pi^2}{3} = 2\sum_{n=1}^{\infty}\frac{1}{n^2},$$

and dividing by 2 gives

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}.$$

This interpretation feels very different from Euler's proof. Here the series arises from the geometry of functions and their oscillations. The answer $\pi^2/6$ is not appearing by accident; it is built into the structure of Fourier analysis itself.

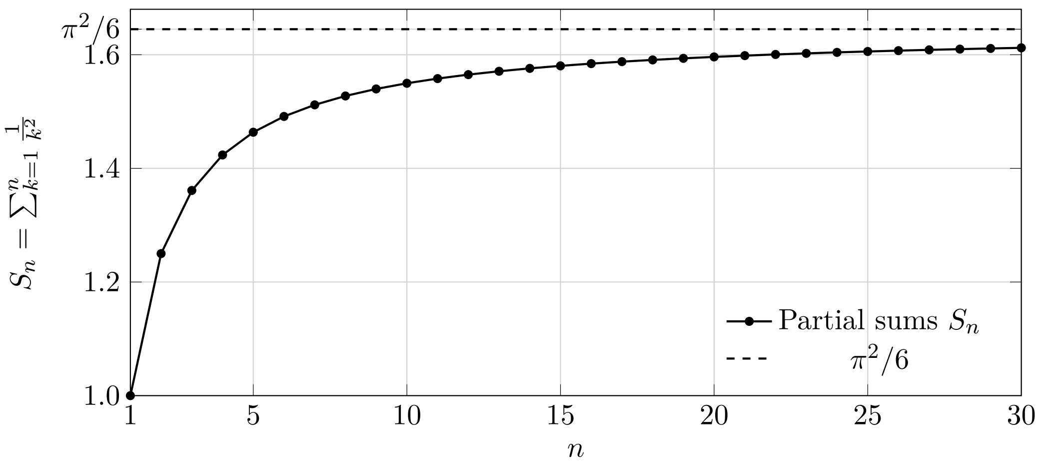

Figure 1: Partial sums of the Basel series approaching the limit $\pi^2/6$. The graph rises quickly at first, then gradually flattens as each new term contributes less than the previous one.

If we plot the partial sums

$$S_n = \sum_{k=1}^{n}\frac{1}{k^2},$$

the graph rises quickly at first and then gradually flattens out. Each new term contributes less than the one before it, causing the curve to approach the horizontal line $y = \pi^2/6$. This makes the convergence visible rather than merely symbolic.

7. The Riemann Zeta Function

The Basel problem is actually the special case $s = 2$ of the **Riemann zeta function**

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}.$$

Euler showed that

$$\zeta(2) = \frac{\pi^2}{6},$$

and he later found similar formulas for other even integers:

$$\zeta(4) = \frac{\pi^4}{90}, \qquad \zeta(6) = \frac{\pi^6}{945}.$$

These identities suggested that the zeta function held much deeper structure than anyone had first expected. Later, Bernhard Riemann extended the zeta function to complex values of s and investigated its zeros. This led to the famous **Riemann Hypothesis**, one of the most important unsolved problems in mathematics [3].

So the Basel problem is not just a solved puzzle. It is the entrance to a much larger mathematical landscape.

8. A Connection to Prime Numbers

Euler also discovered a remarkable product formula:

$$\zeta(s) = \prod_p \frac{1}{1 - p^{-s}},$$

where the product runs over all prime numbers p.

This formula arises from the unique factorisation of integers. Every positive integer can be written as a product of primes in exactly one way. When the product is expanded, it reproduces the entire infinite sum defining $\zeta(s)$.

That means the zeta function links two seemingly different worlds:

- sums over all integers, and
- products over all prime numbers.

This connection is one of the foundations of analytic number theory. It also shows that the Basel problem is tied not only to geometry and analysis, but also to the most fundamental objects in arithmetic: the prime numbers.

9. The Mystery of $\zeta(3)$

Euler found elegant formulas for $\zeta(2)$, $\zeta(4)$, $\zeta(6)$, and the other even values of the zeta function. But the odd values remain much more mysterious.

For example,

$$\zeta(3) = 1 + \frac{1}{8} + \frac{1}{27} + \frac{1}{64} + \cdots .$$

Numerically,

$$\zeta(3) \approx 1.202056.$$

This number is called **Apéry's constant**. In 1979 Roger Apéry proved that it is irrational, but no comparably simple expression involving π is known [3].

This contrast is one of the fascinating features of the zeta function. The even values are beautifully structured; the odd values remain far more elusive. So even after Euler's triumph with the Basel problem, the wider story is not complete. The problem opens a door, but beyond it lie further questions that are still unsolved.

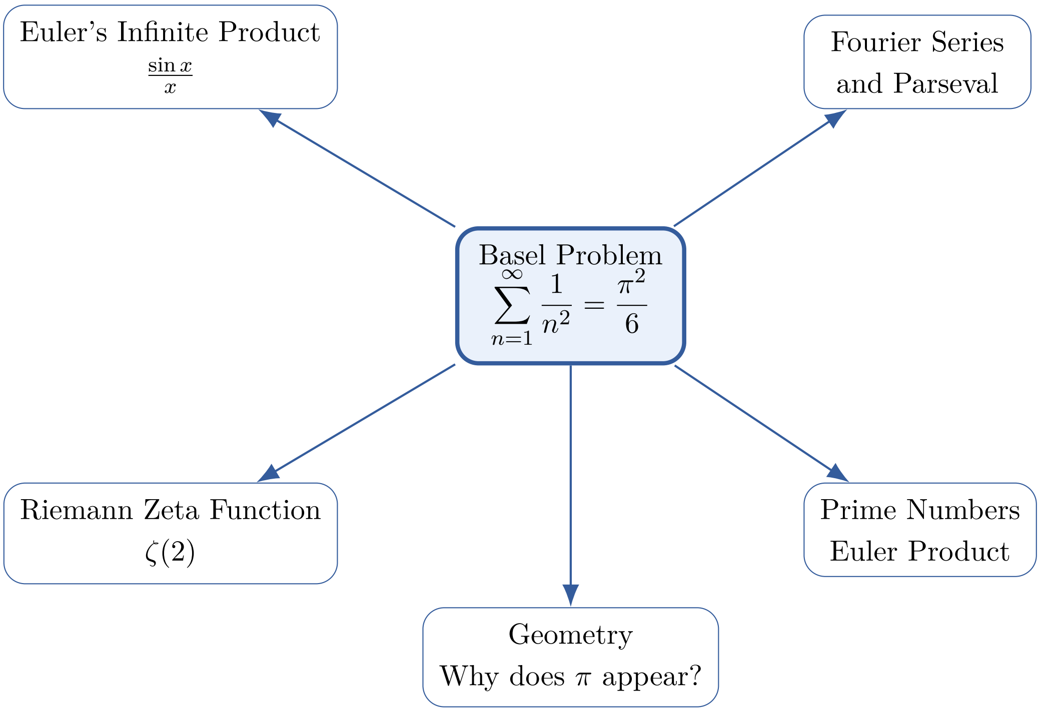

Figure 2: A concept map showing how the Basel problem connects arithmetic, trigonometric functions, Fourier analysis, the zeta function, prime numbers, and geometry.

10. Conclusion

The Basel problem begins with a simple question: what number do the reciprocals of the square numbers add up to?

Euler's solution revealed far more than a numerical answer. It uncovered an unexpected relationship between integers and trigonometric functions. Later developments connected the problem to Fourier analysis, the Riemann zeta function, and the distribution of prime numbers.

The appearance of π in the result is especially striking. A constant associated with circles emerges naturally from a sum involving integers. What looks at first like a problem in arithmetic turns out to be entangled with geometry, analysis, and number theory.

That is what makes the Basel problem so beautiful. It demonstrates one of the deepest features of mathematics: ideas that appear unrelated often turn out to be connected in unexpected and elegant ways. A simple infinite series leads to Fourier waves, prime numbers, and some of the deepest questions in modern mathematics.

What begins as a question about square numbers becomes something much larger: a glimpse of the hidden unity of mathematics itself.

References

[1] Abbott, S. (2015). *Understanding Analysis* (2nd ed.). Springer.

[2] Dunham, W. (1999). *Euler: The Master of Us All*. Mathematical Association of America.

[3] Edwards, H. M. (2001). *Riemann's Zeta Function*. Dover Publications.

[4] Weisstein, E. W. *Basel Problem*. MathWorld — A Wolfram Web Resource. Available at: https://mathworld.wolfram.com/BaselProblem.html

[5] MacTutor History of Mathematics Archive. *Leonhard Euler*. University of St Andrews. Available at: https://mathshistory.st-andrews.ac.uk/Biographies/Euler/