The Infamous Question 6

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1 Introduction

There have been many mathematical problems throughout history that have captivated mathematicians, inspiring them to devote several years, even decades of their life to solving them. The beauty in these problems lies not only in their sheer difficulty, but in the ingenuity they demand and the elegance and harmony they embody. However, it is difficult for non-mathematicians to fully grasp this degree of beauty in maths, as almost all such problems, for example, the Millennium Problems, require significant mathematical background even to be understood. Nonetheless, mathematical Olympiads offer a glimpse into this world, revealing the creativity and elegance that characterises advanced mathematics, while only requiring rudimentary techniques.

At the pinnacle of these mathematical Olympiads, stands the International Mathematical Olympiad. It is, without question, the most difficult maths contest in the world, and each year, every participating country select 6 of their brightest students to represent it. The IMO takes place over two days with 3 problems each day and participants are given 4.5 hours on each day to attempt to solve these challenging questions. Among these seemingly impossible questions, there is one that stands apart in the history of this competition - the 6th problem in the 1988 IMO.

To appreciate how difficult this question is, it is helpful to explore some of its history. At the time it was proposed, nobody on the Australian problem committee could solve it, and since the IMO was held in Australia that year, the question was sent to the 4 most renowned number theorists in Australia. They were given 6 hours to work on this question, and none of them could solve it in time perhaps because the solution of this question essentially required the invention of a new technique, now widely know as "Vieta Jumping". Famously, Terence Tao, one of the greatest minds of our generation, could not solve this question in the competition, and scored only 1/7 (of course, we ignore the fact that he was only 13 at the time). Nevertheless, the question was solved perfectly by 11 students that year.[3]

2 The question the myth the legend

Without further ado, here is the legendary question and its solution:

2.1 Problem (IMO 1988, Question 6).

Let a and b be positive integers such that $ab + 1$ divides $a^2 + b^2$. Show that

$$\frac{a^2 + b^2}{ab + 1}$$

is a perfect square. [1]

2.2 Solution

Let us define

$$k = \frac{a^2 + b^2}{ab + 1}.$$

We must show that k is a perfect square.

We may begin by assuming that k is not a perfect square. Among all such pairs (a, b) satisfying the condition, choose one with minimal value of $a + b$, and without loss of generality assume $a \geq b$. From the definition of k, we have

$$a^2 + b^2 = k(ab + 1).$$

Rewriting this as a quadratic in a,

$$a^2 - kba + (b^2 - k) = 0.$$

We may rewrite this quadratic in terms of x just to see it clearer,

$$x^2 - kbx + (b^2 - k) = 0.$$

Since we know that a is a root to this quadratic, by Vieta's formula, the other root x_2 to thsi quadratic satisfies

$$a + x_2 = kb, \quad ax_2 = b^2 - k.$$

Thus,

$$x_2 = kb - a = \frac{b^2 - k}{a}$$

We now show that x_2 is a positive integer smaller than a. Firstly, we want to show $x_2 > 0$. There 2 cases we need to consider where x_2 is not positive. First is when $x_2 = 0$, which is when $k = b^2$ but we assumed that k is not a square so this is impossible. Second is when $x_2 < 0$, which means that $x_2 \leq -1$ since x_2 is an integer. Thus we have $x_2^2 \geq 1$, and hence

$$x_2^2 - (kb)x_2 + (b^2 - k) \geq 1 + kb + b^2 - k = 1 + b^2 + k(b-1)$$

Therefore when $b > 1$, the equation is larger than 0 and has no real roots, and when $b = 1$, the equation equals 2, which also means it has no real roots, leading to a contradiction and hence $x_2 > 0$. Also,

$$x_2 = \frac{b^2 - k}{a} \leq \frac{a^2 - k}{a} < \frac{a^2}{a} = a$$

thus we also have $x_2 < a$. This means that, (x_2, b) is another pair of positive integers satisfying the same condition:

$$\frac{x_2^2 + b^2}{x_2 b + 1} = k.$$

But now we have $x_2 + b < a + b$, contradicting the minimality of (a, b). Hence, our assumption was false, and k must be a perfect square. $\square$

2.3 Further explanation

The first time I learned the solution, I felt a profound sense of dissatisfaction. The contradiction statement seemed bizarre, how can we just assume the minimality? And what does it prove if we found another smaller pair of solution? The answer to this is an interesting proof called infinite descent. The idea behind it is that if we can keep finding smallest pairs of positive integer solutions to this equation infinitely, as we have shown, then we would eventually have to pass below 1, but again, we have proved x_n is a positive integer.

3 Geometric Interpretation

Another interesting way to look at this question is by considering a hyperbola. By replacing a and b by x and y, the equation could be written as

$$\frac{x^2 + y^2}{xy + 1} = k$$

which can be expressed as

$$x^2 - (ky)x + (y^2 - k) = 0$$

By considering the discriminant of a conic section, we realise that $k > 2$. If $k < 2$, it can only be 1 since k is a positive integer, and 1 is a square so we don't have to worry about this case. Thus we may consider the equation as a hyperbola on the coordinate axis. The famous name of the technique used to solve this question is called "Vieta Jumping", and it gets its name from the equivalent of a "jump" between two points on the hyperbola that share the same y-coordinate.

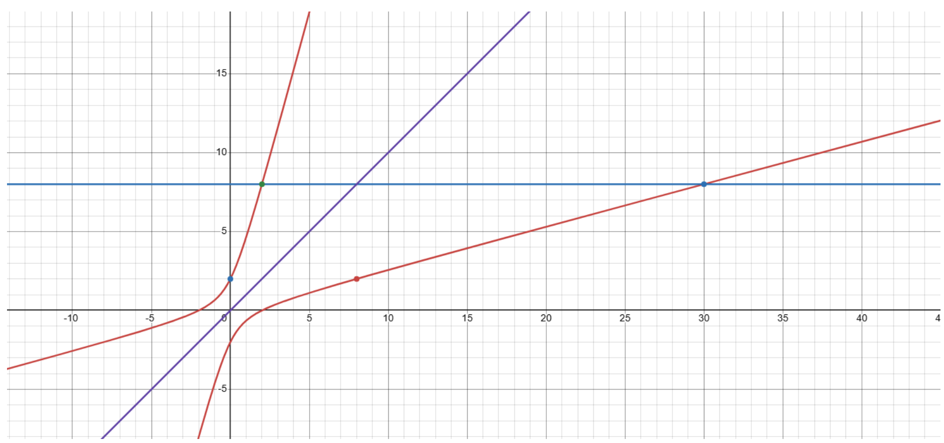

Figure 1

In Figure 1, we have the hyperbola when $k = 4$. The blue line represents the horizontal line $y = 8$, and we may see that it intersects the hyperbola at the point $(30, 8)$ and $(2, 8)$. This demonstrates the jump from the root $x = 30$ to $x = 2$. Since the equation is symmetric, we may reflect the point $(2, 8)$

in the line $y = x$ and obtain another pair of solution, $(8, 2)$. Repeating the same process with the line $y = 2$, we again obtain 2 intersection points, one with a smaller x - coordinate. This illustrates the infinite descent argument because geometrically, each step allows us to go from one pair of solutions to another pair of solutions, with a smaller x - coordinate. This produces a decreasing sequence of positive integers, which we know will eventually end.

4 Further explorations: Markov's Equation

Although "Vieta Jumping" was virtually unknown to competitive mathematics before the 1988 IMO, its underlying mathematical principles had existed for centuries and was used to study Diophantine equations. Another interesting instance where this method comes up is the study of Markov's Equation. Markov's Equation is defined for three variables x, y, z where they are all positive integers and satisfies the equation:

$$x^2 + y^2 + z^2 = 3xyz$$

and "Vieta Jumping" is used to generate solutions to this equation. Firstly, it is evident that the triple $(1, 1, 1)$ is a solution to the equation, and now let us use the same method to discover new solutions to this equation. Although there are 3 variables, we need not worry, let us fix the variable x and treat y and z as constants, we thus have

$$x^2 - (3yz)x + (y^2 + z^2) = 0$$

Let us define the roots of this equation as x_1 and x_2. Using Vieta's formula, we may obtain

$$x_1 + x_2 = 3yz$$

Therefore we have

$$x_2 = 3yz - x_1$$

and since we know that $(1, 1, 1)$ is a solution, let $x_1 = 1, y = 1, z = 1$ we have

$$x_2 = 3 \cdot 1 \cdot 1 - 1 = 2$$

and so $(1, 1, 2)$ is also a solution to this equation. Evidently, we can keep generating solutions to this equation using this method. Because we can choose to fix either x, y or z each time, the solutions to Markov's Equation can be branched out like a tree, and this is famously known as Markov's Tree as illustrated in figure 2,

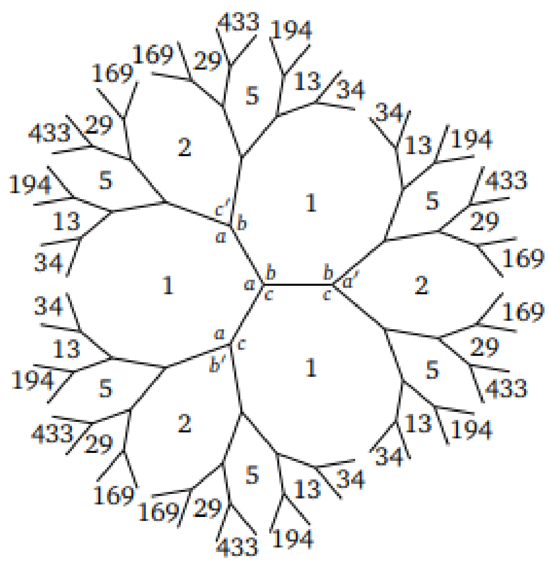

Figure 2

where each node of the tree represents a Markov's Triple. Interestingly, a famous unsolved conjecture called the Markov's Uniqueness Conjecture investigates a property of this equation: the conjecture states "For any given Markov number m, there is exactly one normalized Markov triple (x, y, m) where m is the largest element." The reason why this conjecture is studied so much is because it is linked to other fields of mathematics, for example, in hyperbolic geometry, if the conjecture is proven to be true, it would mean that every simple closed curve of a specific length on a punctured torus is identical under symmetry since the length of the loop is given by the formula

$$L = 2\,\text{arcosh}(3M/2)$$

There are many other fascinating connections to other areas of maths that the Markov Equation has, and it perfectly demonstrates the beauty of the interconnectedness of mathematics.

5 Conclusion

After the IMO in 1988, the technique of "Vieta Jumping" has become extremely well known and has appeared countless of times in various mathematical Olympiads around the world. Here are 2 interesting questions for the readers to attempt if they wish, that can be solved using "Vieta jumping":

1. Let a and b be positive integers. Show that if $4ab-1$ divides $(4a^2-1)^2$, then $a=b$. (IMO 2007 Question 5) [4]

2. Are there infinitely many pairs of positive integers (m,n) such that both m divides n^2+1 and n divides m^2+1. (BMO2 2013 Question 1) [5]

Ultimately, I believe the reason why this specific problem is so famous isn't simply due to its immense difficulty but the elegance of the tool required to solve it. The solution to this question didn't require a new branch of mathematics that has not yet been discovered but required a more symmetric, creative and perhaps simpler way of approaching it.

"The chief forms of beauty are order symmetry and precision, which the mathematical sciences demonstrate to a special degree" - Aristotle.

References

[1] International Mathematical Olympiad. (1988). *Official Results and Problem Set of the 29th IMO*. Canberra, Australia.

[2] Markov, A. A. (1879). *Sur les formes quadratiques binaires indéfinies*. Mathematische Annalen.

[3] Engel, A. (1998). *Problem-Solving Strategies*. Springer-Verlag.

[4] International Mathematical Olympiad. (2007). *Compendium of the 48th IMO: Problem 5*. Hanoi, Vietnam.

[5] British Mathematical Olympiad. (2013). *BMO Round 2: Problem 1*. United Kingdom Mathematics Trust.