# Theta Function & Its Origin

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## Introduction:

We humans were mostly interested in the astronomy since ancient time and desired to explore about planets and our nearest star 'sun'. Since 16th century Kepler given his law of planetary motion, people were interested in calculating the elliptical orbits or circumference of ellipses. In late 17th century Newton and Libniz were successful in developing Calculus. This gives new way of looking at orbits rather than just by geometry.

## Ellipse and Elliptic Integrals:

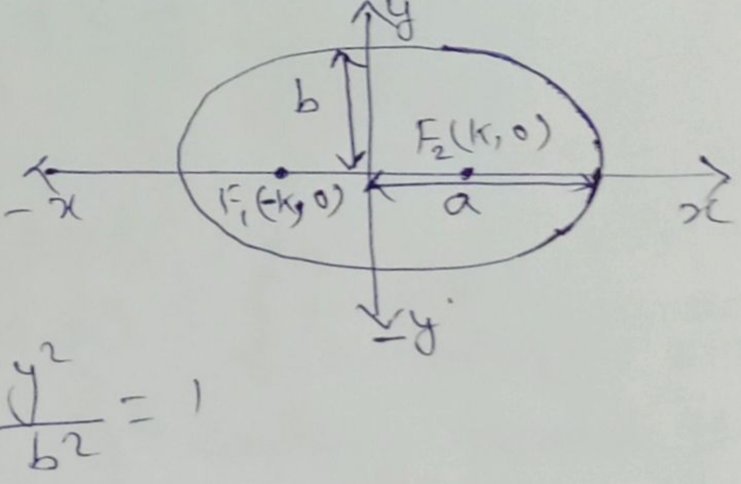
We know that equation of ellipse is given by following

$$\boxed{\frac{x^2}{a^2}+\frac{y^2}{b^2}=1}$$ where eccentricity $k$ is $0<k<1$ and $b^2 = a^2(1-k^2)$

We know that arc length of a curve is given by the following integration:

$$L=\int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Hence we have ellipse and its equation which is shown below:-




$$\Rightarrow \frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\Rightarrow y = \pm \frac{b}{a}\sqrt{a^2-x^2}$$

Differentiating both sides with respect to $x$

$$\Rightarrow y' = \pm \frac{b(-2x)}{2a\sqrt{a^2-x^2}}$$

$$\Rightarrow \frac{dy}{dx} = \pm \frac{-bx}{a\sqrt{a^2-x^2}}$$

Putting the value of $\frac{dy}{dx}$ in L and taking limit of integration from $-a$ to $a$ we get,

$$L = 2\int_{-a}^{a} \sqrt{1+\left(\frac{-bx}{a\sqrt{a^2-x^2}}\right)^2}\, dx$$

Let $x = a\sin\theta$

$dx = a\cos\theta\, d\theta$ and $\theta \in [0; \frac{\pi}{2}]$

$$L = 4\int_0^{\pi/2} \sqrt{1+\left(\frac{-ba\sin\theta}{a\sqrt{a^2-a^2\sin^2\theta}}\right)^2}\; a\cos\theta\, d\theta$$

$$L = 4\int_0^{\pi/2} \sqrt{1+\frac{b^2a^2\sin^2\theta}{a^2(a^2-a^2\sin^2\theta)}}\; a\cos\theta\, d\theta$$

$$L = 4\int_0^{\pi/2} \frac{\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}}{\cancel{a\cos\theta}}\; \cancel{a\cos\theta}\, d\theta$$

$$L = 4\int_0^{\pi/2} \sqrt{a^2(1-\sin^2\theta)+a^2(1-k^2)\sin^2\theta}\; d\theta$$

$$L = 4a\int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\; d\theta$$

Several mathematicians tried to solve this integration but eventually failed for centuries.

Legendre defined three Complete elliptic integrals which are following:-

- First kind

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}}$$

- Second kind

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta}\; d\theta$$

- Third kind

$$\Pi(n,k) = \int_0^{\pi/2} \frac{d\theta}{(1-n^2\sin^2\theta)\sqrt{1-k^2\sin^2\theta}}$$

Abel in 19th century proved that there is no inverse of this elliptic integrals exists in elementary functions and these are doubly periodic functions. This helps another great mind Jacobi to change the way of looking the integrals and use a different approach.

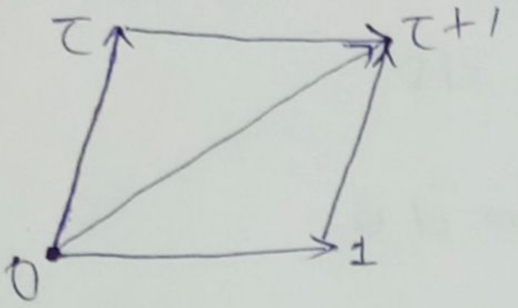
## Jacobi's Theta Function:

Jacobi defined his theta function as following:-

$$\vartheta(z|\tau) = \sum_{n=-\infty}^{\infty} e^{\pi i n^2 \tau + 2\pi i n z}$$

where $z, \tau \in \mathbb{C}$ and $\operatorname{Im}(\tau) > 0$

## The Fundamental Domain:




A fundamental domain refers to a region of the complex variable where the function takes all essentially different values without repetition.

The $\Theta$-function is periodic and quasi-periodic under shifts like $z \to z+1$ and $z \to z+\tau$.

Let $q = e^{\pi i \tau}$ which is also called nome.

→ Shifting the $\vartheta$-function by $z+1$

$$\vartheta(z) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n z}$$

$$\vartheta(z+1) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n(z+1)}$$

$$\vartheta(z+1) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n z} \cdot e^{2\pi i n}$$

$$\vartheta(z+1) = \vartheta(z), \quad \text{since } e^{2\pi i n} = 1 \ \forall\, n \in \mathbb{Z}$$

→ Shifting the $\vartheta$-function by $z+\tau$:

$$\vartheta(z+\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n(z+\tau)}$$

$$\vartheta(z+\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n z + 2\pi i n \tau}$$

$$\vartheta(z+\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i \tau (n^2+2n)} \cdot e^{2\pi i n z} \cdot e^{-\pi i \tau + \pi i \tau}$$

$$\vartheta(z+\tau) = e^{-\pi i \tau} \sum_{n\in\mathbb{Z}} e^{\pi i \tau (n+1)^2} e^{(2\pi i n z)}.$$

Reindexing $m = n+1$

$$\vartheta(z+\tau) = \sum_{m\in\mathbb{Z}} e^{\pi i \tau m^2 + 2\pi i m z + \pi i \tau - \pi i \tau - 2\pi i z}$$

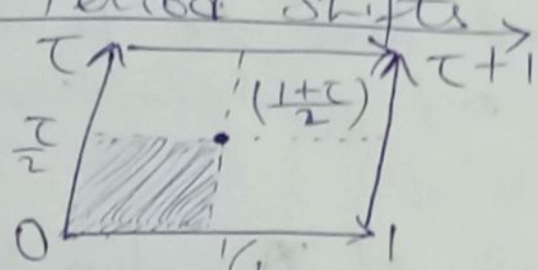
$$\vartheta(z+\tau) = e^{-\pi i \tau - 2\pi i z} \sum_{m\in\mathbb{Z}} e^{\pi i \tau m^2 + 2\pi i m z}$$

$$\vartheta(z+\tau) = e^{-\pi i \tau - 2\pi i z}\, \vartheta(z)$$

Here $\vartheta(z|q) = \vartheta(z)$ is used for simpler notation.

- $e^{-\pi i \tau - 2\pi i z}$ is a quasi factor periodic in $\tau$

↳ <u>Half Period Shifts</u>




Similarly we can get thetas for half period shifts.

① $\vartheta\left(z + \frac{1+\tau}{2}\right) = \vartheta_1(z) = \frac{1}{\sqrt[4]{q}} \sum_{n\in\mathbb{Z}} (-1)^n q^{(n+1/2)^2} e^{2\pi i n z}$

② $\vartheta\left(z + \frac{\tau}{2}\right) = \vartheta_2(z) = \frac{1}{\sqrt[4]{q}} \sum_{n\in\mathbb{Z}} (+1)^n q^{(n+1/2)^2} e^{2\pi i n z}$

③ $\vartheta(z+0) = \vartheta_3(z) = \sum_{n\in\mathbb{Z}} q^{n^2} e^{2\pi i n z}$

④ $\vartheta\left(z+\frac{1}{2}\right) = \vartheta_4(z) = \sum_{n\in\mathbb{Z}} (-1)^n q^{n^2} e^{2\pi i n z}$

- We know that $\theta$-constants satisfies the identity relations :-

$$\vartheta_3^4(0) = \vartheta_2^4(0) + \vartheta_4^4(0)$$

This identity can be written as :

$$1 = \frac{\vartheta_2^4(0)}{\vartheta_3^4(0)} + \frac{\vartheta_4^4(0)}{\vartheta_3^4(0)}$$

↳ Jacobi defines the parameters as modulus

$k^2 = \frac{\vartheta_2^2(0)}{\vartheta_3^2(0)}$ and $\bar{k}^2 = \frac{\vartheta_4^2(0)}{\vartheta_3^2(0)}$ as the Complementary modulus.

Now relation becomes,

$$\bar{k}^2 = 1 - k^2$$

↳ This is orthogonal direction on the torus.

As mentioned earlier that nome is given by
$q = e^{i\pi\tau}$ and also know about modulus
$k$ and $\bar{k}$ complementary modulus.
Hence the ratio of periods satisfies

$$\tau = \frac{iK'(k)}{K(k)}$$

Jacobi defines the complete elliptic integrals as

$$K(k) = \frac{\pi}{2}\vartheta_3^2(0,q), \quad K'(k) = \frac{\pi}{2}\vartheta_4^2(0,q)$$

This shows mathematically how Jacobi theta functions determine the modulus $k$ and how the periods of theta functions are directly expressed in terms of complete elliptic integrals.
Thus the periodic structure of $\vartheta$ - functions encodes the complete elliptic integrals through the modulus $k$ and the period ratio $\tau$.

↳ <u>Functional Equation for $\vartheta$-Function.</u>

We have Jacobi's $\vartheta$ - Function as follows:-

$$\vartheta(z|\tau) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2\tau + 2\pi i n z} \quad \text{where } z,\tau \in \mathbb{C} \text{ and } \operatorname{Im}(\tau) > 0$$

Since $\operatorname{Im}(\tau) > 0 \Rightarrow$ rapid converges hence we can use Poisson Summation formula.

$$\sum_{n\in\mathbb{Z}} f(n) = \sum_{k\in\mathbb{Z}} \hat{f}(k) \quad \{\text{Fourier Transform}\}$$

$$\vartheta(0|i\tau) = \sum_{n\in\mathbb{Z}} e^{-\pi n^2\tau}$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2\tau} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\pi y^2\tau} e^{-2\pi i k y}\, dy$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2\tau} = \sum_{k\in\mathbb{Z}} \int_{\mathbb{R}} e^{-\pi\tau\left(y^2 + \frac{2iyk}{\tau}\right)}\, dy$$

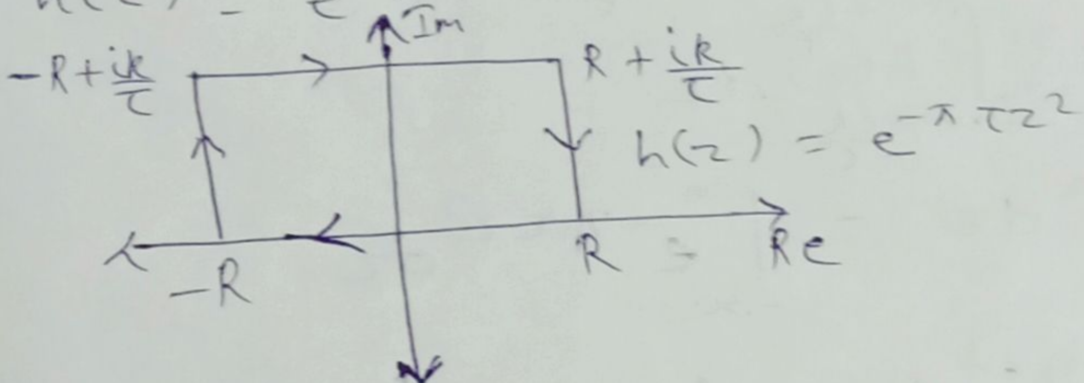
$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi n^2\tau} = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{\tau}} \int_{\mathbb{R}} e^{-\pi\tau\left(y + \frac{ik}{\tau}\right)^2}\, dy$$

Let $z = y + \frac{ik}{\tau} \Rightarrow dz = dy$

$$\sum_{n\in\mathbb{Z}} e^{-\pi\tau n^2} = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{\tau}} \int_{-\infty+\frac{ik}{\tau}}^{\infty+\frac{ik}{\tau}} e^{-\pi\tau z^2}\, dz$$

Let $I_0 = \int_{-\infty+\frac{ik}{\tau}}^{\infty+\frac{ik}{\tau}} e^{-\pi\tau z^2}\, dz$

Let $h(z) = e^{-\pi\tau z^2}$




Hence we have,

$$\oint h(z)dz = \int_{R}^{-R} h(z)dz + \int_{-R}^{-R+\frac{ik}{\tau}} h(z)dz + \int_{-R+\frac{ik}{\tau}}^{R+\frac{ik}{\tau}} h(z)dz$$
$$+ \int_{R+\frac{ik}{\tau}}^{R} h(z)dz$$

By Cauchy's Residue Theorem,

$\oint h(z)dz = 0$ [since there no poles, no residue]

$$\Rightarrow \int_{R}^{-R} h(z)dz = -\left(\int_{-R}^{-R+\frac{ik}{\tau}} h(z)dz + \int_{R+\frac{ik}{\tau}}^{R} h(z)dz\right) - \int_{-R+\frac{ik}{\tau}}^{R+\frac{ik}{\tau}} h(z)dz$$

$$\Rightarrow \int_{-R}^{R} h(z)dz = \left(\int_{-R}^{-R+\frac{ik}{\tau}} h(z)dz + \int_{R+\frac{ik}{\tau}}^{R} h(z)dz\right) + \int_{-R+\frac{ik}{\tau}}^{R+\frac{ik}{\tau}} h(z)dz$$

Let $I_1 = \int_{-R}^{-R+\frac{ik}{\tau}} h(z)dz$

$$I_1 = \lim_{R\to\infty} \int_{-R}^{-R+\frac{ik}{\tau}} e^{-\pi\tau z^2} dz \quad ; \text{ let } z = -R+iy$$
$$dz = i\,dy$$

$$I_1 = \lim_{R\to\infty} \int_{0}^{k/\tau} e^{-\pi\tau(R^2-y^2-2Riy)}\, i\,dy$$

$I_1 = 0$

Similarily let $I_2 = \int_{R+\frac{ik}{\tau}}^{R} h(z)dz$

$\Rightarrow I_2 = 0$

We get, $\int_{-R}^{R} h(z)\,dz = \int_{-R+\frac{ik}{\tau}}^{R+\frac{ik}{\tau}} h(z)\,dz$

Taking limit as $R \to \infty$ we get,

$$\lim_{R\to\infty}\int_{-R}^{R} h(z)\,dz = \lim_{R\to\infty}\int_{-R+\frac{ik}{\tau}}^{R+\frac{ik}{\tau}} h(z)\,dz$$

$$\Rightarrow \int_{-\infty}^{\infty} h(z)\,dz = \int_{-\infty+\frac{ik}{\tau}}^{\infty+\frac{ik}{\tau}} h(z)\,dz$$

$$\Rightarrow \int_{-\infty+\frac{ik}{\tau}}^{\infty+\frac{ik}{\tau}} e^{-\pi\tau z^2}\,dz = \int_{-\infty}^{\infty} e^{-\pi\tau z^2}\,dz$$

Hence we get,

$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi\tau z^2} = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{\tau}} \int_{-\infty}^{\infty} e^{-\pi\tau z^2}\,dz$$

$$\Rightarrow \sum_{n\in\mathbb{Z}} e^{-\pi\tau z^2} = \sum_{k\in\mathbb{Z}} e^{-\frac{\pi k^2}{\tau}} \frac{1}{\sqrt{\tau}}$$

$$\boxed{\vartheta(0|i\tau) = \frac{1}{\sqrt{\tau}}\vartheta(0|i\tau^{-1})}$$

## Applications of Theta Function:

1. Jacobi derived the product identity for theta function which is following: It helps in partition theory and combinatorics:

$$\prod_{n=0}^{\infty}(1-q^{2n+2})(1+zq^{2n+1})\left(1+\frac{q^{2n+1}}{z}\right) = \sum_{n\in\mathbb{Z}} q^{n^2} z^n$$

Valid for $|q|<1$ and $z \neq 0$.

2. Theta function is also used in study of heat equations.

3. Theta function is used to in quantum physics and string theory.

4. It also helps in study of modular forms.

5. Later theta function's version helps in study of black holes.

↳ Generalization and Ramanujan Theta Functions

Ramanujan defined the theta function as follows:

$$f(a,b) = \sum_{n\in\mathbb{Z}} a^{\frac{n(n+1)}{2}} b^{\frac{n(n-1)}{2}} \quad \text{for } |ab|<1$$

↳ Integral Representation:

$$f(a,b) = 1 + \int_0^\infty \frac{2a e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} \left[\frac{1 - a\sqrt{ab}\cosh(\sqrt{\log ab}\, t)}{a^3 b - 2a\sqrt{ab}\cosh(\sqrt{\log ab}\, t)+1}\right] dt$$

$$+ \int_0^\infty \frac{2b e^{-\frac{1}{2}t^2}}{\sqrt{2\pi}} \left[\frac{1 - b\sqrt{ab}\cosh(\sqrt{\log ab}\, t)}{ab^3 - 2b\sqrt{ab}\cosh(\sqrt{\log ab}\, t)+1}\right] dt$$

Ramanujan's this generalization helps in study of more powerful things such as in string theory. Later with help of theta function astrophysticts ~~in~~ studies black holes.

↳ Conclusions:-

We can see a long history behind origin of theta function and its beautiful world which only come when we changed our prespective of solving the problem and looking the world.

Thank you for giving this wonderful opportunities and for your precious time.