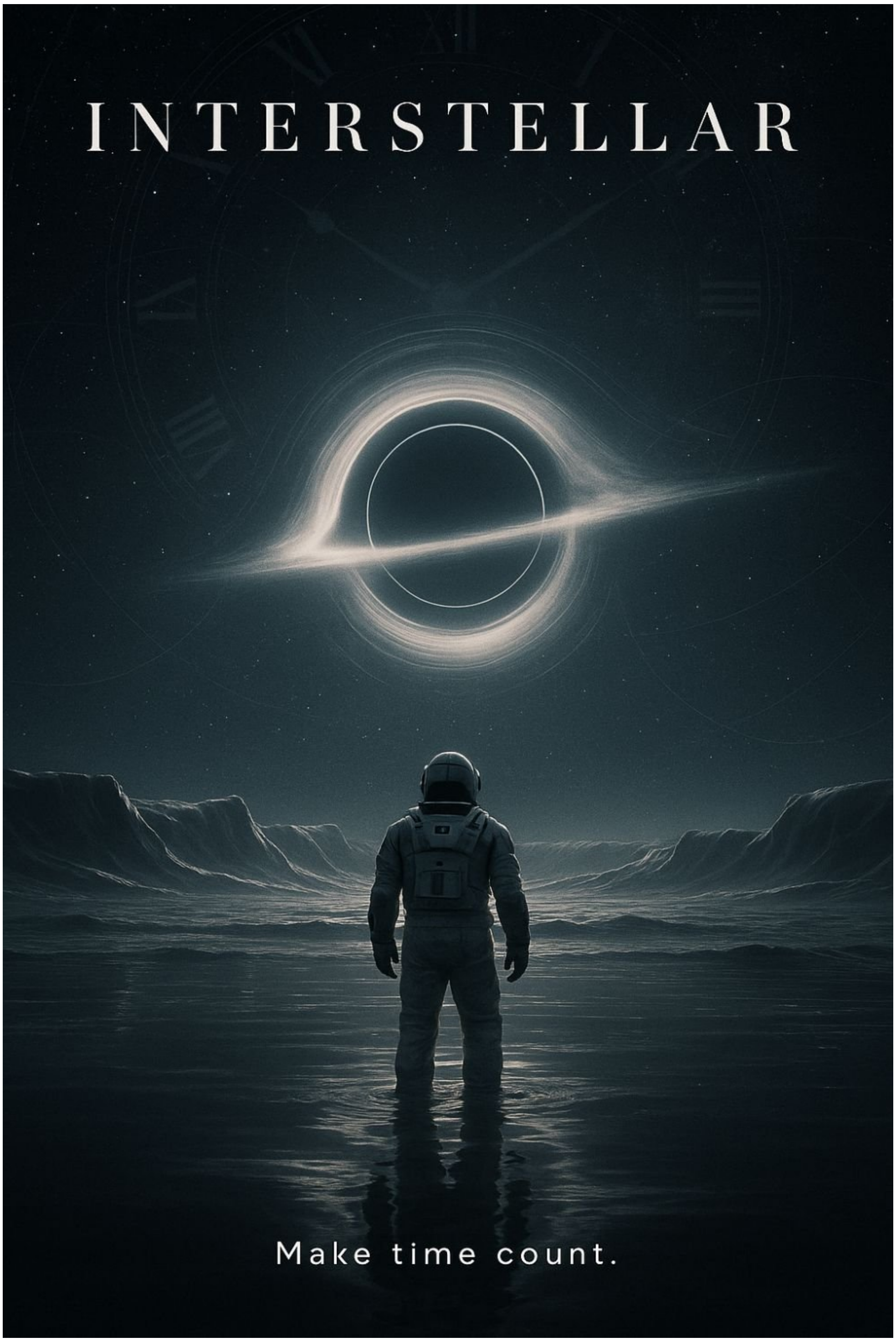

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Introduction

In the sci-fi film Interstellar, a team of astronauts, including Joseph Cooper and Dr. Amelia Brand travels through a wormhole near Saturn in search of a habitable planet known as “Miller’s Planet”. Early in the film, Cooper explains a startling concept: on this planet, one hour is equivalent to seven years on Earth.

At first glance, this idea seems impossible, almost fictional. How could time pass so differently in two places within the same universe? However, this phenomenon is not just science fiction. It is based on a real scientific concept known as time dilation, first described by Albert Einstein in his theory of relativity. The purpose of this paper is to explore the mathematical equation of time dilation, using both gravity and velocity.

Overview (Miller's Planet)

Before we can explore the mathematical concept, we must first understand why time works differently on Miller's planet. As mentioned, one hour on this planet is equivalent to seven years on Earth, this is due to the immense gravitational pull of the nearby black hole, Gargantua.

According to Albert Einstein's theory of relativity, gravity is not just a force but a curvature of spacetime itself. The stronger the gravitational field, the more spacetime is curved, causing time to pass more slowly relative to regions with weaker gravity. In addition to gravitational effects, time can also be affected by velocity, as objects moving at very high speeds experience time more slowly compared to those at rest.Because of this intense curvature of spacetime and speed, time passes significantly slower there compared to Earth.

Time Dilation

Time dilation is a physical phenomenon where time moves slower for objects travelling at a high velocity or experiencing intense gravity compared to a stationary object. According to Einstein's theory, time is not absolute.Time dilation describes how moving clocks tick slower compared to a stationary observer's clock. Time can vary based on an object's speed (Special relativity) and gravity (General relativity).

Using Variables

Time dilation can be described using variables that represent time measured in different reference points. Where t represents the proper time, which time is measured by an observer in the same point of the event, and t' represents the dilated time, which is measured by an observer in a different point. These two values are not equal, as time alone can't be measured but needs physical properties such as velocity and gravitational strength.

When comparing two observers, one located in a region of weak gravity or low velocity and another in a region of strong gravity or high velocity, the observer in the more extreme conditions will measure a shorter time dilation. This means that $t'<t$, demonstrating that time passes more slowly under stronger gravitational fields and higher velocities. Therefore, time dilation can be understood as a comparison between two measurements of time that vary depending on the observer's point of reference.

Gravitational Time Dilation

The gravitational time dilation equation relates the time experienced by an observer near a massive object to the time experienced by an observer farther away.

$$t' = t\sqrt{1 - \frac{2GM}{rc^2}}$$

t'= time measured closer to the massive object (dilated time)

t = time measured by a distant observer (proper time)

G = Gravitational consent (6.6743 × 10^{-11} m3 kg^{-1} s^{-2})

M = Mass of object

r = Radial distance from center

c = Speed of light (299,792,458 m/s)

Mathematically, the equation shows that as the mass M increases, the value inside the square root becomes smaller, which causes t' to decrease. Similarly, as the distance r decreases, the fraction becomes larger, further reducing the value of t'. This means that stronger gravitational fields, caused by larger masses or closer distances, result in greater time dilation. As a result, time passes more slowly near massive objects compared to regions with weaker gravitational influence.

Velocity Time Dilation

The velocity time dilation equation depends on the relative motion between two observers.

$$t' = \frac{t}{\sqrt{1 - \frac{v^2}{c^2}}}$$

t'= time measured by an observer moving at high velocity (dilated time)

t = time measured by a distant observer (proper time)

v = velocity

c = the speed of light

Mathematically, the equation demonstrates that as velocity increases and approaches the speed of light,the value inside the square root decreases, which causes t' to increase. This means that higher velocity, results in greater time dilation. As a result,time slows down for the moving object compared to an observer at rest.

Comparison

In gravitational time dilation, the expression inside the square root depends on both the distance r from the mass and the mass M itself, showing how gravity influences time. In contrast, velocity time dilation compares the velocity v of an object to the speed of light c. In both cases, the equations approach a limit under extreme conditions, and time slows down for the object experiencing stronger gravity or higher velocity.

Application

Miller’s Planet is located in an extremely strong gravitational field near the black hole Gargantua. To analyze the effect on time, the gravitational time dilation equation is used

$$\frac{t'}{t} = \sqrt{1 - \frac{2GM}{rc^2}}$$

Where

G = 6.67 * 10^{-11}

c = 3.0 * 10^{8} ----> c2 = 9.0 * 10^{16}

M ≈ 2 * 10^{38} kg

$r_s \approx \frac{2GM}{c^2}$

r = 2.0 * 10^{11}

Step 1: compute the Schwarzschild radius

$$r_s \simeq \frac{2GM}{c^2} \simeq \frac{2(6.67\times10^{-11})\times(2\times10^{38})}{9.0\times10^{16}}$$

Step 2: simplify numerator

$$2 \times 6.67 \times 2 = 26.68$$

$$10^{-11} \times 10^{38} = 10^{27}$$

Step 3 : Slove r_s

$$r_s = \frac{26.68 \times 10^{27}}{9 \times 10^{10}}$$

$$r_s \approx 2.96 \times 10^{11}\ m$$

Step 4: choosing an orbital value

r = 3.0 * 10^{11}

Step 5 : Solve rc^2

$$rc^2 = (3.0 \times 10^{11})(9.0 \times 10^{16}) = 2.7 \times 10^{28}$$

Step 6 : Form fraction

$$\frac{2GM}{rc^2} = \frac{2.668 \times 10^{28}}{2.7 \times 10^{28}} \approx 0.988$$

Step 7 : Substitute to equation

$$\frac{t'}{t} = \sqrt{1-0.988}$$

$$\frac{t'}{t} = \sqrt{0.012} \approx 0.11$$

Step 8: conclude

The calculated ration $\frac{t'}{t} \approx 0.11$indicates that the time measured near the massive object runs at approximately 11% of the rate experienced by the distance observer. This result follows directly from the gravitational time dilation equation, where the term $\frac{2GM}{rc2}$ depends on both the mass of the object and the radial distance from its center.

To better understand the context of this result, the Schwarzschild radius r_s= $\frac{2GM}{c2}$ was computed as approximately 2.96×10^{11}. The chosen orbital radius r=3.0×10^{11} m r = 3.0 $\times10^{11}$lies just outside this, meaning the system is within a strong gravitational field but still within the range where the equation remains valid.

As the radial distance r approaches the Schwarzschild radius $r_{s,}$,the fraction $\frac{2GM}{rc2}$ approaches 1, causing the value inside the square root to approach zero. This leads to increasingly strong time dilation. This explains how, under more extreme conditions like those experienced on Miller's Planet, time could slow to the extent that one hour corresponds to several years for a distant observer.

Thus, the mathematical model demonstrates that the extreme time difference observed on Miller's Planet is a direct consequence of its proximity to a massive black hole, where gravitational effects dominate and drastically reduce the rate at which time passes.

Conclusion

Both gravitational and velocity time dilation describe how time changes for an observer, but they do so in fundamentally different ways. Gravitational time dilation relates the time experienced near a massive object to the time experienced farther away, while velocity time dilation depends on the relative motion between observers. This demonstrates that time is not always uniform, but instead depends on factors such as gravitational strength and relative velocity.

These relationships are mathematically predictable, as each is defined by a precise equation that allows accurate calculation of time under different conditions. By substituting known values into these equations, it becomes possible to model real-world scenarios and observe how changes in gravity and velocity affect the passage of time.

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