The Exploration of Order in Van der Waerden Theorem.

Part One: Philosophical basis and nature of Ramsey Theory.

Mathematics is a subject that the general population views as an attempt to find certain solutions to numeric issues. But to persons who enter into the higher provinces of combinatorics, mathematics is a subject not of calculation, but of the discovery of inevitable truths. Among the most far-reaching of these facts is one which Van der Waerden has expressed in his Theorem. The given theorem is one of the pillars of Ramsey Theory, which is a branch of mathematics that could be reduced to one impressive statement: there is no such thing as total disorder. As a system grows to a large scale, some sorts of patterns must arise, whether it is desired or not. This discovery changed perceptions of mathematicians about randomness and turned it into a condition of pure chaos which is subjected to unseen, structural laws.

In order to appreciate the value of the contribution of Van der Waerden, we must first examine the environment that gave birth to the contribution. At the beginning of the twentieth century, mathematicians started to study the nature of infinite sets and the behavior of finite subdivisions of the infinite sets. It was a deceptively simple question: given the set of all positive integers, and breaking them into a finite number of subsets (usually pictured as coloring the numbers with a few different colors), do you know that at least one of the subsets will include a sequence of numbers that are spaced apart by a perfect gap? This series is referred to as an arithmetic progression. As an example, the numbers five, ten and fifteen are an arithmetic progression of length three since the difference between the consecutive terms of the progression is always five.

Prior to Van der Waerden giving his demonstration in the late nineteen twenties, many were theorizing that such a pattern could be evaded provided that the colouring was complicated enough. It was possible to imagine a very elaborate pattern of colors that slides and varies so as that three numbers of the same color never form a straight line with the same distance between them. But this intuition was proved false by the theorem. However ingenious the colouring, and however many colours may be employed, so long as there are a finite number of colours, however many colours may be used, there is a mathematical certainty that a monochromatic arithmetic progression of any length one might wish to have can be obtained.

This confidence brings about an intriguing philosophical twist. It implies that structure is not always imposed on a system by an external force but structure is a property of large systems. We tend to give coincidence or design a pattern in our daily life. When we observe a chain of events which appear to be in an order, we suppose that it is not there in vain. The Theorem of van der Waerden informs us that in the number world, the reason is merely the magnitude of the system itself. When you have sufficient data, sufficient figures, or sufficient terms in a series, the laws of logic have it that they must at one time or another take on a recognizable direction.

It is this inevitable nature that makes Ramsey Theory so intriguing. It considers mathematics as the hunt after the Ramsey Numbers or Van der Waerden Numbers, the precise point where chaos has to yield to order. That is a threshold of nine in a progression of length three and two colors. It follows that in case you paint the numbers 1 to 9 in two colours, you are sure to have three or more numbers of one colour in an arithmetic sequence. With eight numbers you might get away with it. This point of tipping is a logical constant.

The implications of this theorem go way beyond the chalk board. It comes close to the essence of information. The contemporary age of large data is one that is always seeking meaning in the noise. The Theorem of van der Waerden is like a caution and a compass: it informs us that there are signals which are not in reality signals, but are the natural consequence of the presence of a great deal of noise. Assuming that a data set is large enough, it will also contain all possible patterns merely because it has exhausted its capacity to be random. Such knowledge is very important to scientists and researchers who need to know the difference between a discovery made that is meaningful and a mathematical necessity.

Moreover, the theorem accentuates the beauty of the integers. We tend to believe that numbers are separate entities but Van der Waerden demonstrated that they are a closely bonded web of relations. These possible progressions connect each and every number together. We are merely accentuating that which was already there by deciding to color them. The pattern is not made by the theorem, but only shows that the pattern is a permanent tenant of the number system.

When we pass to the philosophical implication of the theorem into the mechanical reality of it, we start to perceive the fantastic resourcefulness it takes to demonstrate such a statement. Suspicion that there must be a pattern, is one thing; to demonstrate that it must not exist, is another. This demands some rigour of logic that stretches the limits of human thought. The evidence is not based on a mere observation but on a gigantic, structural induction which accumulates itself to the end until the conclusion is irresistible. In the following passages we shall discuss the particular machinery of this demonstration, and how it is able to overcome the infinite.

Part Two: Architecture of the Proof and Logic of Induction.

It takes a change in our way of thinking about mathematical structures to move the beauty of the Van der Waerden Theorem, in the philosophical sense, to the actual mechanical proof. The evidence is a masterpiece in the terms of what is called double induction. We cannot just go to a single case to demonstrate that there must be a pattern to any length and any number of colors. Rather, we need to establish a rational ladder in which one step leads to another and ultimately to the infinite. This part examines the technical approach which enables mathematicians to assert that order is a mathematical certitude.

The concept of the base case is the first step to understand the proof. In any argument that is inductive, you have to begin with a truth that is so simple that it is indubitable. For this theorem, the base case is a progression of length two. When you take a number of colors indeed and take enough integers, you are sure to get a pair of integers of the same color. This fact makes it trivially true, as any two integers of the same colour are technically part of an arithmetic progression whose length is two. The difficulty, though, is in extrapolating a known fact of short sequences to a certain truth of sequences of arbitrary length.

To overcome this obstacle, Van der Waerden adopted an ingenious method on the basis of blocks of integers. He did not count one number at a time, but put them in huge blocks. He then considered each of the distinct methods of coloring a block as a single new color. An example would be to consider coloring blocks of three numbers with red and blue, then there are only eight ways to color the blocks. We can apply the inductive hypothesis to these blocks by treating each of these eight patterns as a super-color. This enables us to locate a pattern of blocks that have an independent pattern.

This gives rise to the finding of what is referred to as a fan of progressions. Consider a number of various arithmetic progressions, each of which is seeking the same last number to complete the sequence with. All these progressions are such that they all began at various locations but would all reach the same integer should they each take an additional step. By the inductive step we may demonstrate that there must be at least one of these existing progressions in a large set which is already monochromatic. It implies that all the numbers in that particular order are already of the same color, such as, green.

The reasoning then drives a conclusion, depending on the color of that last, common landing point. When that spot is green we have the green progression complete, and we have our pattern. When that spot is painted in any other color, e.g., yellow, it does not matter, as the system is constructed to take into consideration any possible color choice. With several layers of these fans we have a case in which any conceivable decision on what the final number shall be will in any case complete a sequence of at least one of the colours concerned. That is why this is so powerful a theorem, it makes the freedom of choice a trap where all the routes will come to the same orderly solution.

One essential element of this evidence is that it does not give a small or easy to check number of integers. The size of the set needed to ensure the pattern grows at an astronomical rate is also due to the fact that each step of the induction must consider entire colorings as new units. It is here that we come to the Ackermann function and other forms of hyper-operation sequences. This demonstration that the number of integers required exists, and demonstrates also that this number is so great that it is beyond ordinary notation.

The genius of this method is, that it makes the existence of large patterns to be forced by the existence of small ones. It is a recursive process. By learning that we can always find a pattern of length three, we are providing the means by which to show that we can always find a pattern of length four. Knowing that we can find a pattern of length four, we can prove that five, six, etc., and so on, until we have exhaustively considered every possible length of pattern. This is the way to raise the truth up to the next level, characteristic of modern combinatorics.

In addition, this evidence shows that the pigeonhole principle is significant in a high dimensional space. The pigeonhole principle is that when you have more items than containers, then at least one container must contain more than one item. This simple idea is the one which Van der Waerden proves by patterns and colorings. When there are only so many ways to color a block of numbers, and you have enough blocks, then (as of the patterns in those blocks) you must eventually have repeats. It is this repetition which feeds the inductive engine.

In short the second part of the voyage in this theorem shows that the demonstration is no one aha! but a fortress of logic built up. It employs the notion of blocks and super-color to transform a disorderly coloring issue into a systematic search of symmetry. Van der Waerden ruled out the possibility of chaos by demonstrating that all the conceivable final colors in a sequence are such that they complete a progression. In the last section we shall examine the heritage of this demonstration and how it has branched out to the most intricate realms of prime numbers and beyond.

Part Three: The Legacy of Structure and the Prime Frontier.

The last stage of comprehending Van der Waerden Theorem is to look beyond the first proof and understand how it changed the face of modern mathematics. Although the initial work by itself was a monumental work in its own right, the real strength of it is how it served as a seed to even more grandiose discoveries. The section discusses the way in which the theorem began as an interesting fact about colored integers, and grew to become a powerful tool in the study of the most mysterious sequences of all, such as the prime numbers.

The work of Endre Szemelyi was one of the earliest large expansions of this logic. As Van der Waerden was thinking about coloring integers, Szemeredi questioned himself what would happen were we to now cease worrying about colors and instead consider the density of a set. He showed that you do not have to divide the integers into groups in order to arrive at order. Any set of integers large enough (that is, containing a positive upper density) must contain arithmetic progressions of any size. This was an enormous step forward since it shifted the discussion out of a subjective attribution of colors to a objective fact of the numbers themselves. It demonstrated that internal structure is guaranteed by the presence of thickness in a numerical set.

The reverberations of this finding were to extend ultimately to the most renowned series of them all: the prime numbers. The primes were considered the perfect specimen of pseudo randomness and this was the case centuries ago. They seem to grow out of the number line without any apparent pattern or noticeable spacing. But basing their work on Van der Waerden and Szemeredi, Ben Green and Terence Tao notoriously established in two thousand and four that the prime numbers have arithmetic progressions of all lengths. This implies that somewhere in the infinite expanse of the primes, one will find ten equally-spaced primes in sequence, or 100, or one million. It was a breaking point in history and in the history of number theory in the sense that it demonstrated that even the most random-seeming of our number system is governed by the laws of inevitable order.

Outside the theoretical world, it is the legacy of this theorem that manifests in the manner we deal with complexity. It has had an impact on computer science, in the area of algorithmic information theory. With an insight into the fact that some of the patterns are mathematically imposed, scientists will be able to conceive better to what is really meant by the terms information and redundancy. When a pattern is required by law, the existence of a pattern does not provide as unique information as a pattern that occurs voluntarily or a pattern that is a result of particular data. The difference is critical in data compression and the entropy of digital systems.

The theorem is also a reminder of the human and machine limits of computation. The Van der Waerden numbers increase so rapidly as discussed earlier that they cannot be computed at all even at small numbers. This forms an interesting science border: we know with certainty that a structure exists, but we cannot physically find it. This underscores a type of non-constructive knowledge that undermines our wish to view and touch any mathematical reality. It compels us to use the innocence of reason as opposed to the mightiness of our computers.

When we sum up what we have observed in this excursion we find that the Van der Waerden Theorem is not merely a demonstration concerning colored dots on a line. It is a deep utterance concerning the universe. It infers that the universe at the larger scale is not a disorder tangle but a well-organized masterpiece. In the way of the galaxies, in the discharge of the neurons of the brain, in the order of the integers, we are perpetually confronted with the same principle: when you have plenty of anything you will discover a law.

We learned in the work of van der Waerden that chaos is a transitional phenomenon that only occurs in the absence of sufficient searching. As soon as the system reaches a large size, the noise starts to synchronize, and the chaos starts to dance to a regular, predictable beat. This is the beautifulest example of human thought ever, as it is the one which gives us the transition of the finite to the infinite, and is the one which assures us that, however far we may venture in the unknown, we shall always have within reach that soothing influence of order. The mathematical demonstration is that nothing in the great plan of things is ever really lost to chaos.