Modelling and Optimizing the tennis serve

Introduction

Being an avid self-taught tennis player of 6 years, I have always been stumped by the serve at the beginning of the match, wherein I have more often than not failed to deliver a convincing serve, leading to many unfortunate losses, such as in the figure below where I **lost by 1 point in a final match** due to failing to deliver the serve twice, leading to an untimely penalty.

Figure 1: Losing the tennis match because I failed to deliver a serve

Back then, I always blamed it on several other factor such as the weather, sweaty palms, the ball and even my racquet. However, upon much reflection, I realized that **my serves had always been inconsistent, in particular my angle of contact** with the ball.

However, with many years of suffering numerous losses and having tasted one too many humble pies due to my inadequacies in the serve, I decided it was time to take accountability and instead focus on modelling and delivering the best serve I physically could deliver. **It was time to yield the power of mathematics to optimize my release angle to help me develop an "unreturnable serve"!**

Research Aim

My Research Aim for this Mathematics Exploration Essay is **"Modelling and optimizing the tennis serve using 3D projectile motion and the Lambert W Function".**

Background

In serving **diagonally** from the baseline, an optimal serve is that which lands the ball on the **corner end of the service line** on the opponent's side of the court[1], as indicated by the red arrowhead in figure 2 below. For the exploration, the **serve is to be taken from the centre of the baseline from which I am most comfortable serving**, as represented by the blue dot.

---

[1] Patrick Mouratoglou. (n.d.). *Tennis serving technique*. Tennis Serving Technique | Mouratoglou Tennis Academy. https://www.mouratoglou.com/en/coaching-corner/tennis-technique/serving-technique/

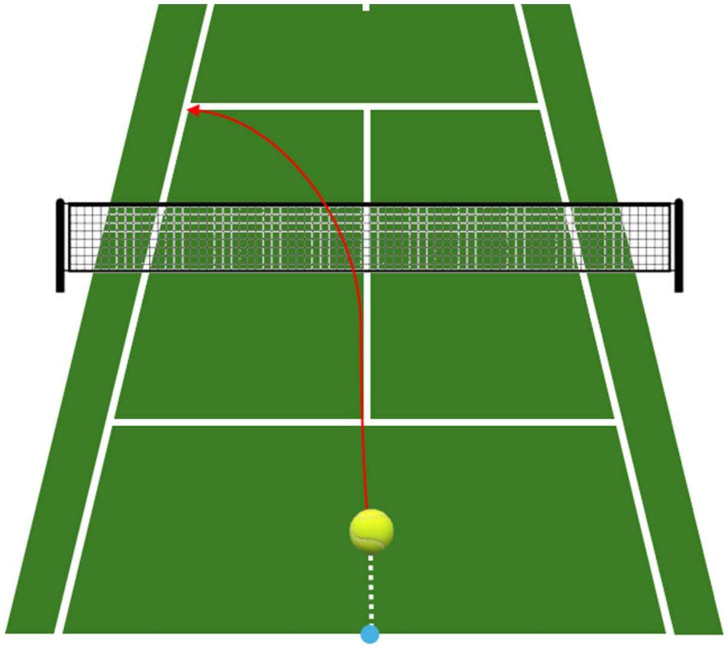


Figure 2: Trajectory of a tennis ball upon service [2] [3]

To maximize the effectiveness of the shot, **the time taken** for the ball to land on the opponent's side of the court **should be minimized** such that there is minimal time for the opponent to react and return the serve. Thus, the final objective of the calculations would be to minimize this value.

a. Dimensions, Formulas and Derivations

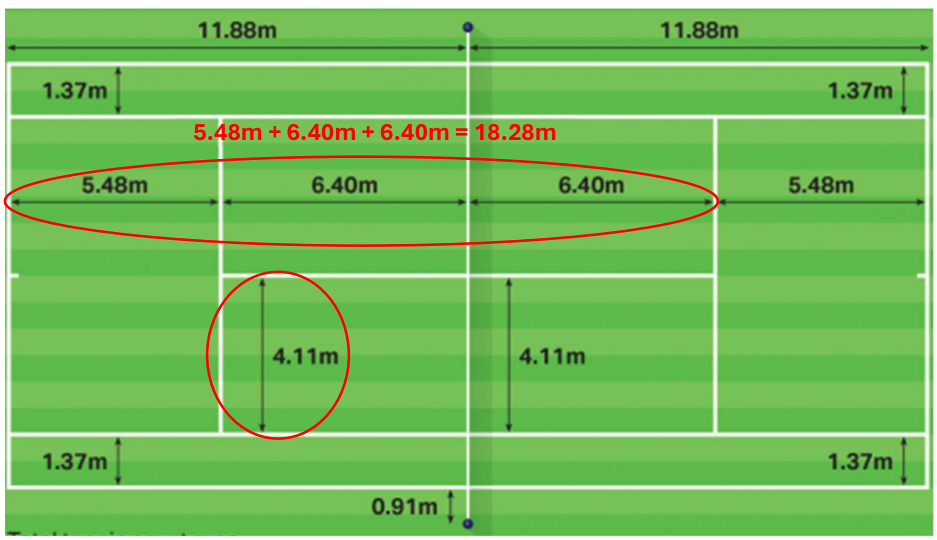


Figure 3: Dimensions of a standard tennis court [4]

Figure 4: Height of point of contact with ball

Using the dimensions of a standard tennis court (Figure 3) and my point of contact with the tennis ball when serving (Figure 4, measured via a measuring tape), a diagram with displacement vectors of the respective axes $\vec{x}$, $\vec{y}$ and $\vec{z}$ were obtained. Thus, the resultant displacement vector $\vec{s}$ can be expressed as such:

$$\vec{s} = \begin{pmatrix} \vec{x} \\ \vec{y} \\ \vec{z} \end{pmatrix} = \begin{pmatrix} 18.28 \\ -3.00 \\ 4.11 \end{pmatrix}$$

[2] Kontur-Vid, Sutana, S., Pialhovik, RomanSotola, Bigmouse108, ser_igor, Annetka, Bortonia, Rahadian, V., Yehezkel, Y., MegaShabanov, Yotto, KeithBishop, Axel_wolf, Karelkart, TheArtist, AltoClassic, Tomacco, INDECcraft, … Archive, N. D. (n.d.). Tennis net stock illustrations. iStock. https://www.istockphoto.com/illustrations/tennis-net

[3] Tennis ball images - free download on Freepik. Freepik. (n.d.). https://www.freepik.com/free-photos-vectors/tennis-ball

[4] By. (n.d.). Tennis Court Dimensions & Size. ISP. https://www.harrodsport.com/advice-and-guides/tennis-court-dimensions

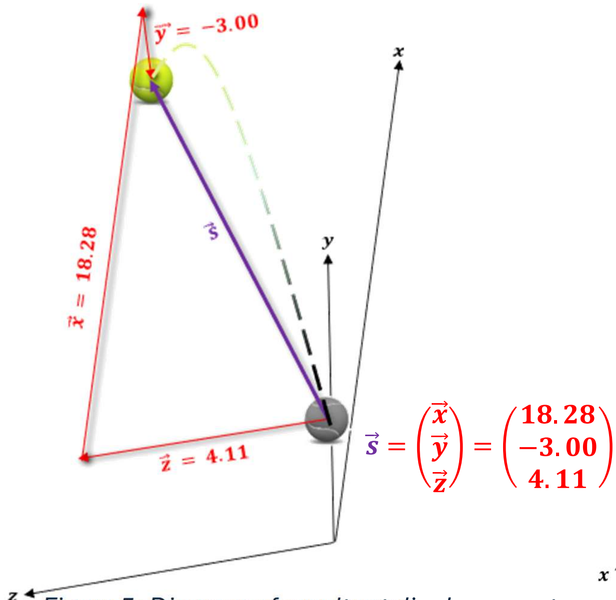


Figure 5: Diagram of resultant displacement

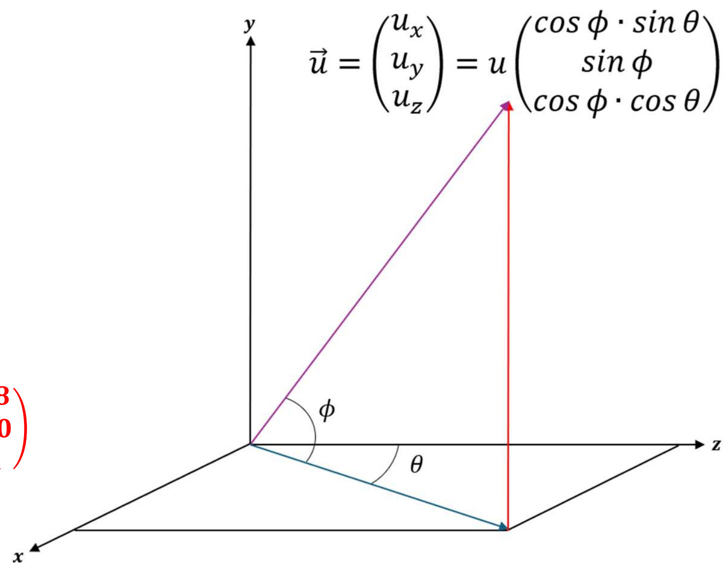


Figure 6: Diagram of the initial velocity of the tennis ball

Next, a simplified 3D cartesian plane will be used to derive the individual components of initial velocity of the tennis ball, wherein the initial velocity u will be broken down into its individual components in the x, y and z axes ($u_x,\ u_y,\ u_z$). Utilizing basic trigonometry with reference to the two angles, $\boldsymbol{\theta}$ (angle projected onto horizontal plane) and $\boldsymbol{\phi}$ (angle between vertical and horizontal plane), an expression for the constituents of the initial velocity $\boldsymbol{u}$ can be acquired through expansion, factorization and the identity $(\sin\theta)^2 + (\cos\theta)^2 \equiv 1$

$$u = \sqrt{{u_x}^2 + {u_y}^2 + {u_z}^2} = \sqrt{u^2 \cdot (\cos^2\theta\ \cdot \cos^2\phi + \cos^2\theta \cdot \sin^2\phi + \sin^2\theta)}$$

$$= \sqrt{u^2 \cdot (\cos^2\theta\,(\cos^2\phi\ + \sin^2\phi) +\ \sin^2\theta)} = \sqrt{u^2 \cdot (\cos^2\theta\,(\cos^2\phi\ + \sin^2\phi) +\ \sin^2\theta)}$$

$$= \sqrt{u^2 \cdot (\cos^2\theta\,(1) +\ \sin^2\theta)} = \sqrt{u^2 \cdot (\cos^2\theta + \sin^2\theta)} = \sqrt{u^2 \cdot 1} = u$$

Next, before finding an expression for the equation of motion for the relative velocity of all three components, the following notations will be used for the exploration (subscripts correspond to respective axis of motion):

Acceleration, $\boldsymbol{a}$ $(\boldsymbol{ms^{-2}})$: a_x; a_y; a_z	Velocity, $\boldsymbol{v}$ $(\boldsymbol{ms^{-1}})$: v_x; v_y; v_z	Displacement, $\boldsymbol{s}$ $(\boldsymbol{m})$: x ; y; z
Time $(\boldsymbol{s})$: t	Gravitational acceleration constant $(\boldsymbol{ms^{-2}})$: g, experimentally determined to be 9.80665 [5]	

<u>Relating kinematics equations through derivatives:</u>

$$a = \frac{d}{dt}(v) = \frac{d}{dt}\left(\frac{d}{dt}(s)\right) = \frac{d^2 s}{dt^2}$$

Displacement is a vector quantity (with both direction and magnitude) that measures the straight-line distance between two points. Velocity is defined as the rate of change of displacement, hence differentiating displacement with respect to time will yield the expression for velocity, which is hence the 1st derivative of displacement with respect to time. Likewise, acceleration is defined as the rate of change of velocity, hence differentiating velocity with respect to time will yield the expression for acceleration, which is hence the 2nd derivative of displacement with respect to time.

b. The Lambert W Function

The Lambert W Function, otherwise known as the product log, is most commonly defined as the inverse of the function $f(x) = xe^x$ and can be plotted via the equation $x = ye^y$. Generally speaking, for an arbitrary function or constant $g(x)$, the Lambert W function can also be expressed as such:

---

[5] National Institute of Standards and Technology. (n.d.). Standard gravity [Value]. National Institute of Standards and Technology. https://physics.nist.gov/cgi-bin/cuu/Value?gn

$$W\left(g(x)e^{g(x)}\right) = g(x)$$

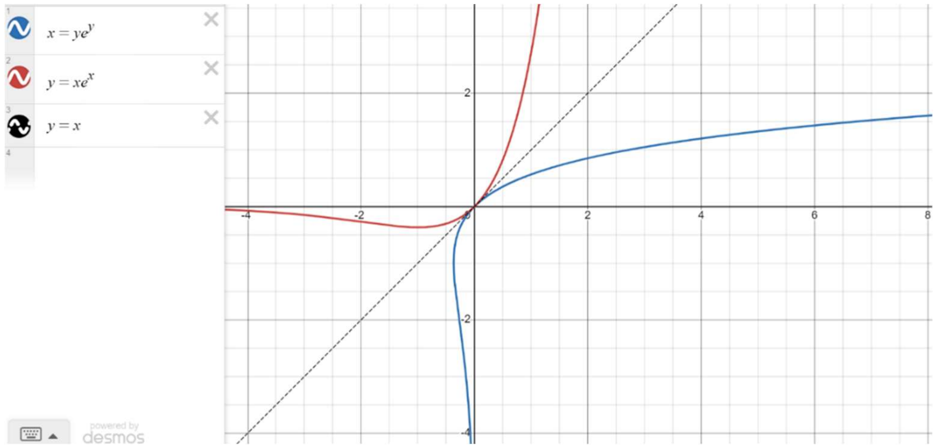


Figure 7: The Lambert W Function plotted onto Desmos (The blue graph is the W Lambert function)

The function provides an elegant, closed-form solution ω for the equation of the form $a\omega + b + ce^{d\omega} = 0$, where $\omega = j(x)$, an arbitrary function or constant, while a, b, c, d are arbitrary constants.

$$a\omega + b = -ce^{d\omega}$$
$$(a\omega + b)e^{-d\omega} = -c$$
$$(a\omega + b)e^{-d\omega} = -c$$

Multiplying both sides by $-\frac{d}{a}$:

$$\left(-d\omega - \frac{db}{a}\right)e^{-d\omega} = \frac{cd}{a}$$

Multiplying both sides by $e^{-\frac{db}{a}}$:

$$\left(-d\omega - \frac{db}{a}\right)e^{-d\omega} \cdot e^{-\frac{db}{a}} = \frac{cd}{a} \cdot e^{-\frac{db}{a}}$$
$$\left(-d\omega - \frac{db}{a}\right)e^{-d\omega - \frac{db}{a}} = \frac{cd}{a} \cdot e^{-\frac{db}{a}}$$

Applying the Lambert W function:

$$W\left(\left(-d\omega - \frac{db}{a}\right)e^{-d\omega - \frac{db}{a}}\right) = W\left(\frac{cd}{a} \cdot e^{-\frac{db}{a}}\right)$$
$$-d\omega - \frac{db}{a} = W\left(\frac{cd}{a} \cdot e^{-\frac{db}{a}}\right)$$

Isolating the ω term:

$$-d\omega = W\left(\frac{cd}{a} \cdot e^{-\frac{db}{a}}\right) + \frac{db}{a}$$
$$\omega = -\frac{1}{d}W\left(\frac{cd}{a} \cdot e^{-\frac{db}{a}}\right) - \frac{b}{a}, \qquad a, d \neq 0 \;---\; (1)$$

This equation is critical in solving equations that involve both a linear and exponential term in the equation such as that in the form $a\omega + b + ce^{d\omega}$, as a closed form solution to these equations can only be acquired through the use of the Lambert W Function. Hence, the function will prove to be of great use in later calculations due to the exponential and linear properties of the equations of motion when the effects of both gravity and drag are considered.

Air resistance and wind (drag):

Accurate calculations cannot be made without integrating the forces of nature into play. In reality, when modelling 3D projectile motion, **air resistance**, otherwise known as **drag**, must be included in the calculations to account for the **resistive force that opposes the motion of the projectile,** resulting in the **deceleration** of the projectile. Additionally, the **downward acceleration experienced by the gravity** of magnitude **g** must also be incorporated into the net downward acceleration experienced by the tennis ball for the vertical component $\boldsymbol{a_y}$.

Furthermore, **air resistance and wind are interdependent** as the resistive force is determined by the relative velocity between the projectile and the air, thus the calculations will take both factors into consideration by relating the **air resistance** experienced by the tennis ball will to the velocity of the tennis ball via a constant of proportionality **k**, as shown in figure 8 below:

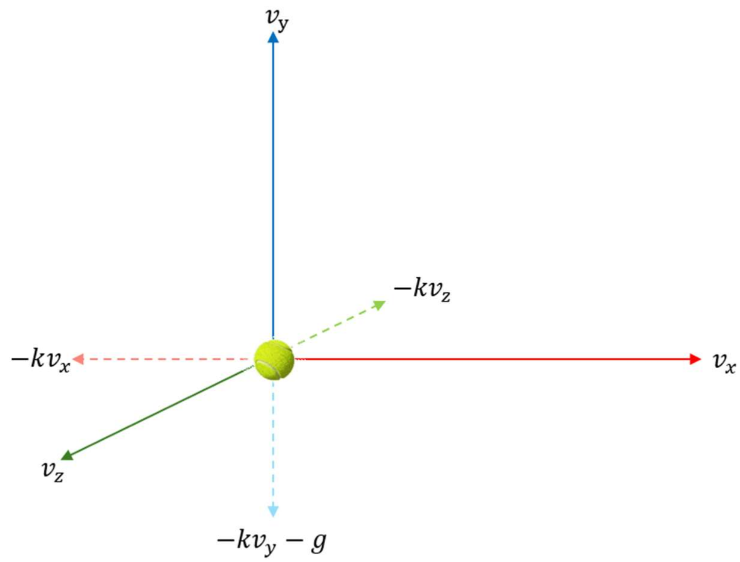


Figure 8: Tennis ball projectile motion with air resistance [6]

Using the formula for force (Newton's second law) and finding the expression for the net deceleration:

$$F_{resistance} = ma = -kv\ ;\ a\ =\ -\frac{k}{m}v;\ where\ m\ is\ the\ mass\ of\ the\ tennis\ ball$$

Hence, using the above expression, we arrive at the following differential equations:

$$a_x = -\frac{k}{m}v_x;\ \boldsymbol{\frac{d}{dt}\left(\frac{dx}{dt}\right) = -\frac{k}{m}\cdot\frac{dx}{dt}}\ ---\ (2)\ ;\ a_z = \frac{k}{m}v_z\ ;\ \boldsymbol{\frac{d}{dt}\left(\frac{dz}{dt}\right) = -\frac{k}{m}\cdot\frac{dz}{dt}}\ ---\ (3)$$

$$a_y = -g\ -\frac{k}{m}v_y\ ;\ \boldsymbol{\frac{d}{dt}\left(\frac{dy}{dt}\right)\ =\ -g\ -\ \frac{k}{m}\cdot\frac{dy}{dt}}\ ---\ (4)$$

<u>Horizontal axes (x and z):</u>

Manipulate (2) and (3) into the form $\boldsymbol{\frac{\frac{d}{dt}(f)}{f}}$ by dividing both sides of the equation by the velocity:

$$\frac{\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\frac{dx}{dt}} = -\frac{k}{m}\ ;\ \frac{\frac{d}{dt}\left(\frac{dz}{dt}\right)}{\frac{dz}{dt}} = -\frac{k}{m}$$

Integrating both sides of the equation with respect to time using $\int\frac{\frac{d}{dt}(f)}{f}\ =\ ln(f)\ +\ C$:

$$\int\frac{\frac{d}{dt}\left(\frac{dx}{dt}\right)}{\frac{dx}{dt}}\,dt = -\int\frac{k}{m}dt\ ;\ \int\frac{\frac{d}{dt}\left(\frac{dz}{dt}\right)}{\frac{dz}{dt}}\,dt = -\int\frac{k}{m}dt$$

Simplifying the constants of the expression:

$$ln\left(\frac{dx}{dt}\right)\ = -\frac{k}{m}t\ +\ C_4\ ;\ ln\left(\frac{dz}{dt}\right)\ = -\frac{k}{m}t\ +\ C_5$$

Exponentiating both sides of the equation:

$$e^{ln\left(\frac{dx}{dt}\right)}\ = e^{-\frac{k}{m}t + C_4}\ ;\ e^{ln\left(\frac{dz}{dt}\right)}\ = e^{-\frac{k}{m}t + C_5}$$

Simplifying the equation using the identity $e^{ln(f(x))}\ =\ f(x)$ and simplifying the constant:

$$\frac{dx}{dt}\ = e^{-\frac{k}{m}t + C_4}\ ;\ \frac{dz}{dt} = e^{-\frac{k}{m}t + C_5}$$

[6] Tennis ball images - free download on Freepik. Freepik. (n.d.). https://www.freepik.com/free-photos-vectors/tennis-ball

$$\frac{dx}{dt} = e^{C_4}(e^{-\frac{k}{m}t})\ ;\ \frac{dz}{dt} = e^{C_5}(e^{-\frac{k}{m}t})$$
$$\frac{dx}{dt} = C_6(e^{-\frac{k}{m}t})\ ;\ \frac{dz}{dt} = C_7(e^{-\frac{k}{m}t})$$

Evaluating constants $\boldsymbol{C_6}$ and $\boldsymbol{C_7}$ by evaluating the initial velocities at $\boldsymbol{t} = 0$:

$$C_6(e^{-\frac{k}{m}(0)}) = C_6 = u_x = u\,(\cos\phi \cdot \sin\theta) \quad ;\ C_7(e^{-\frac{k}{m}(0)}) = C_7 = u_z = u\,(\cos\phi \cdot \cos\theta)$$

Integrating both expressions for velocity with respect to time to find displacement:

$$\vec{x} = \int \left(\frac{dx}{dt}\right) dt = u\,(\cos\phi \cdot \sin\theta) \int e^{-\frac{k}{m}t}\,dt\ ; \qquad \vec{z} = \int \left(\frac{dz}{dt}\right) dt = u\,(\cos\phi \cdot \cos\theta) \int e^{-\frac{k}{m}t}\,dt$$

$$= u\,(\cos\phi \cdot \sin\theta)\left(-\frac{e^{-\frac{k}{m}t}}{\left(\frac{k}{m}\right)}\right) + C_8 \qquad\qquad = u\,(\cos\phi \cdot \cos\theta)\left(-\frac{e^{-kt}}{\left(\frac{k}{m}\right)}\right) + C_9$$

Finding the constant in the integral by evaluating the displacement at $\boldsymbol{t} = 0$:

$$u\,(\cos\phi \cdot \sin\theta)\left(-\frac{e^{-\frac{k}{m}(0)}}{\left(\frac{k}{m}\right)}\right) + C_8 = 0;\ u\,(\cos\phi \cdot \cos\theta)\left(-\frac{e^{-\frac{k}{m}(0)}}{\left(\frac{k}{m}\right)}\right) + C_9 = 0$$

$$u\,(\cos\phi \cdot \sin\theta)\,(-\frac{m}{k}) + C_8 = 0;\ (\cos\phi \cdot \cos\theta)\,(-\frac{m}{k}) + C_9 = 0$$

$$C_8 = (\frac{m}{k})u\,(\cos\phi \cdot \sin\theta)\,;\ C_9 = (\frac{m}{k})(\cos\phi \cdot \cos\theta)$$

Hence,

$$\vec{x} = u\,(\cos\phi \cdot \sin\theta)\left(-\frac{e^{-\frac{k}{m}t}}{\left(\frac{k}{m}\right)}\right) + \left(\frac{m}{k}\right)u\,(\cos\phi \cdot \sin\theta) = \left(\frac{m}{k}\right)u\,(\cos\phi \cdot \sin\theta)(1 - e^{-\frac{k}{m}t}) \quad --- \ (5)$$

$$\vec{z} = u\,(\cos\phi \cdot \cos\theta)\left(-\frac{e^{-\frac{k}{m}t}}{\left(\frac{k}{m}\right)}\right) + \left(\frac{m}{k}\right)(\cos\phi \cdot \cos\theta) = \left(\frac{m}{k}\right)u\,(\cos\phi \cdot \cos\theta)(1 - e^{-\frac{k}{m}t}) --- (6)$$

<u>Vertical axis (y):</u>

Taking the reciprocal of both sides of equation (4):

$$\frac{d\left(\frac{dy}{dt}\right)}{dt} = -(g + \frac{k}{m}\cdot\frac{dy}{dt})$$
$$\frac{dt}{d\left(\frac{dy}{dt}\right)} = -\frac{1}{(g + \frac{k}{m}\cdot\frac{dy}{dt})}$$

Multiplying both sides of the equation by $\boldsymbol{\frac{k}{m}}$ in order to utilize u – substitution:

$$\frac{\frac{k}{m}dt}{d\left(\frac{dy}{dt}\right)} = -\frac{\frac{k}{m}}{(g + \frac{k}{m}\cdot\frac{dy}{dt})}$$

Multiplying both terms by $\boldsymbol{d}\left(\boldsymbol{\frac{dy}{dt}}\right)$ to prepare for integration with respect to both time and velocity:

$$\frac{k}{m}dt = -\frac{\frac{k}{m}}{(g + \frac{k}{m}\cdot\frac{dy}{dt})}d\left(\frac{dy}{dt}\right)$$

Integrating both sides of the equation with respect to time and velocity (with u – substitution):

$$\int \frac{k}{m}dt = -\int \frac{\frac{k}{m}}{(g + \frac{k}{m}\cdot\frac{dy}{dt})}d\left(\frac{dy}{dt}\right)$$

$$u = g + \frac{k}{m}\cdot\frac{dy}{dt};\ du = \frac{k}{m}d\left(\frac{dy}{dt}\right);\ \frac{du}{\left(\frac{k}{m}\right)} = d\left(\frac{dy}{dt}\right)$$

$$\int \frac{k}{m}dt = \int \frac{1}{u}du$$

$$-\frac{k}{m}t + C_{10} = ln\,(u) = ln(g + \frac{k}{m}\cdot\frac{dy}{dt})$$

Exponentiating both sides of the equation:

$$e^{-\frac{k}{m}t + C_{10}} = e^{ln(g + \frac{k}{m}\cdot\frac{dy}{dt})} = g + \frac{k}{m}\cdot\frac{dy}{dt}$$

Solving for the constant $\boldsymbol{C_{11}}$ in the expression of the velocity by evaluating the initial velocity at $\boldsymbol{t} = 0$:

$$e^{-\frac{k}{m}t + C_{10}} = C_{11}(e^{-\frac{k}{m}t}) = g + \frac{k}{m}\cdot\frac{dy}{dt}$$

$$C_{11}(e^{-k(0)}) = C_{11} = g + \frac{k}{m}\cdot u\,sin\,\phi$$

Substituting the $\boldsymbol{C_{11}}$ expression and isolating the $\frac{\boldsymbol{dy}}{\boldsymbol{dt}}$ term in the equation:

$$g + \frac{k}{m}\cdot\frac{dy}{dt} = (\,g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot(e^{-\frac{k}{m}t})$$

$$\frac{dy}{dt} = \frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (e^{-\frac{k}{m}t}) - g}{\left(\frac{k}{m}\right)} = \frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (e^{-\frac{k}{m}t})}{\left(\frac{k}{m}\right)} - \frac{mg}{k}$$

Integrating $\frac{\boldsymbol{dy}}{\boldsymbol{dt}}$ (velocity) with respect to time to find displacement:

$$\vec{y} = \int\left(\frac{dy}{dt}\right)dt = \int\left(\frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (e^{-\frac{k}{m}t})}{\left(\frac{k}{m}\right)} - \frac{mg}{k}\right)dt$$

$$= -\frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (e^{-\frac{k}{m}t})}{\left(\frac{k}{m}\right)^2} - \frac{mg}{k}t + C_{12}$$

Finding the constant $\boldsymbol{C_{12}}$ in the integral by evaluating the displacement at $\boldsymbol{t} = 0$:

$$-\frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (e^{-\frac{k}{m}(0)})}{\left(\frac{k}{m}\right)^2} - \frac{g}{k}(0) + C_{12} = 0$$

$$C_{12} = \frac{g + \frac{k}{m}\cdot u\,sin\,\phi}{\left(\frac{k}{m}\right)^2}$$

Hence,

$$\vec{y} = -\frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (e^{-\frac{k}{m}t})}{\left(\frac{k}{m}\right)^2} - \frac{mg}{k}t + \frac{g + \frac{k}{m}\cdot u\,sin\,\phi}{\left(\frac{k}{m}\right)^2}$$

$$\vec{y} = \frac{(g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (1 - e^{-\frac{k}{m}t}) - \frac{k}{m}\cdot gt}{\left(\frac{k}{m}\right)^2} = \left(\frac{m}{k}\right)^2\left((g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (1 - e^{-\frac{k}{m}t}) - \frac{k}{m}\cdot gt\right) --- (7)$$

Henceforth, an expression for the 3D displacement $\vec{\boldsymbol{s}}$ for the tennis ball in consideration of air resistance and wind is obtained and equated to the aforementioned displacement vector:

$$\vec{s} = \begin{pmatrix}\vec{x}\\ \vec{y}\\ \vec{z}\end{pmatrix} = \begin{pmatrix}\left(\frac{m}{k}\right)u\,(cos\,\phi\cdot sin\,\theta)(1 - e^{-\frac{k}{m}t})\\ \left(\frac{m}{k}\right)^2\left((g + \frac{k}{m}\cdot u\,sin\,\phi)\cdot\ (1 - e^{-\frac{k}{m}t}) - \frac{k}{m}\cdot gt\right)\\ \left(\frac{m}{k}\right)u\,(cos\,\phi\cdot cos\,\theta)(1 - e^{-\frac{k}{m}t})\end{pmatrix} = \begin{pmatrix}18.28\\ -3.00\\ 4.11\end{pmatrix}$$

Modelling the trajectory of the tennis ball:

In order to compare the above equations of motion to the experimental trajectory of the tennis ball, we must first express the vertical displacement $\vec{\boldsymbol{y}}$ in terms of the horizontal displacements ($\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{z}}$) of the tennis ball. This will be accomplished via substitution and manipulation of equations (5), (6) and (7).

Noticing that all three equations have the same term $(1 - e^{-\frac{k}{m}t})$, hence manipulating equations (5) and (6) to express $\vec{x}$ and $\vec{z}$ in terms of $(1 - e^{-\frac{k}{m}t})$:

$$\frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)} = (1 - e^{-\frac{k}{m}t}) \quad --- (8)\ ; \quad \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)} = (1 - e^{-\frac{k}{m}t}) \quad --- (9)$$

Expressing time $\boldsymbol{t}$ in terms of $\vec{x}$ and $\vec{z}$ by manipulating equations (8) and (9):

$$e^{-\frac{k}{m}t} = 1 + \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)}\ ;\ e^{-\frac{k}{m}t} = 1 + \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)}$$

Applying the natural logarithm $\boldsymbol{ln}$ to both sides of the equation:

$$-\frac{k}{m}t = ln\left|1 + \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)}\right|\ ;\ -\frac{k}{m}t = ln\left|1 + \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)}\right|$$

Multiplying both sides of the equation by $-\frac{\boldsymbol{m}}{\boldsymbol{k}}$ to isolate the $\boldsymbol{t}$ term, forming an expression in terms of $\vec{x}$ and $\vec{z}$:

$$t = -\frac{m}{k} ln\left|1 + \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)}\right| \quad --- (10)\ ;\ t = -\frac{m}{k} ln\left|1 + \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)}\right| \quad --- (11)$$

Substituting (8), (9), (10) and (11) into equation (7) to express $\vec{y}$ as a function of $\vec{x}$ and $\vec{z}$:

$$\vec{y}(\vec{x}) = \left(\frac{m}{k}\right)^2 \left((g + \frac{k}{m} \cdot u \sin\phi) \cdot \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)} - \frac{k}{m} \cdot g \left(-\frac{m}{k} ln\left|1 + \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)}\right| \right) \right)$$

$$\vec{y}(\vec{z}) = \left(\frac{m}{k}\right)^2 \left((g + \frac{k}{m} \cdot u \sin\phi) \cdot \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)} - \frac{k}{m} \cdot g \left(-\frac{m}{k} ln\left|1 + \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)}\right| \right) \right)$$

Expanding out all the terms in the equation for $\vec{y}$ as a function of $\vec{x}$ and $\vec{z}$:

$$\vec{y}(\vec{x}) = \frac{g\left(\frac{m}{k}\right)^2 \vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)} + \left(\frac{m}{k}\right)^2 \cdot \frac{\frac{k}{m} \cdot u \sin\phi\ \vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)} - \left(\frac{m}{k}\right)^2 \cdot \frac{k}{m} \cdot \left(-\frac{m}{k}\right) g \cdot ln\left|1 + \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)}\right|$$

$$\vec{y}(\vec{z}) = \frac{g\left(\frac{m}{k}\right)^2 \vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)} + \left(\frac{m}{k}\right)^2 \cdot \frac{\frac{k}{m} \cdot u \sin\phi\ \vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)} - \left(\frac{m}{k}\right)^2 \cdot \frac{k}{m} \cdot \left(-\frac{m}{k}\right) g \cdot ln\left|1 + \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)}\right|$$

Simplifying all the terms in the equation for $\vec{y}$ as a function of $\vec{x}$ and $\vec{z}$:

$$\vec{y}(\vec{x}) = \frac{g\left(\frac{m}{k}\right)\vec{x}}{u\,(\cos\phi \cdot \sin\theta)} + \frac{\tan\phi}{\sin\theta}\,\vec{x} + \left(\frac{m}{k}\right)^2 \cdot g \cdot ln\left|1 + \frac{\vec{x}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)}\right| \quad --- (12)$$

$$\vec{y}(\vec{z}) = \frac{g\left(\frac{m}{k}\right)\vec{z}}{u\,(\cos\phi \cdot \cos\theta)} + \frac{\tan\phi}{\cos\theta}\,\vec{z} + \left(\frac{m}{k}\right)^2 \cdot g \cdot ln\left|1 + \frac{\vec{z}}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)}\right| \quad --- (13)$$

To demonstrate the effect of the constant of proportionality of air resistance $\boldsymbol{k}$ on the trajectory of the tennis ball, the following steps were taken:

1. In order to visualize the effect of $\boldsymbol{k}$ on the trajectory of the tennis ball, the graphs for $\vec{y}$ as a function of $\vec{x}$ will be plotted with varying values of $\boldsymbol{k}$, which demonstrates the effect of how a greater drag force (perhaps due to wind or the temperature of the air) would impact the motion of the tennis ball.
2. Due to the numerous variables $\boldsymbol{m}, \boldsymbol{u}, \boldsymbol{\phi}, \boldsymbol{\theta}$ in the equation for $\vec{y}$ as a function of $\vec{x}$, we will first set these values to be constants, which is acceptable as we are only exploring the effect of variating $\boldsymbol{k}$ for now.
3. The mass of the tennis ball $\boldsymbol{m}$ was set to an arbitrary constant of 0.050 kg.
4. The initial velocity of the tennis ball $\boldsymbol{u}$ was set to an arbitrary value of 50 m s^{-1}.

5. The angle between the vertical plane and the horizontal plane $\boldsymbol{\phi}$ was set to an arbitrary value of -2°.
6. The angle projected onto horizontal plane $\boldsymbol{\theta}$ was set to an arbitrary value of 20°.
7. 5 graphs for $\vec{y}$ as a function of $\vec{x}$ with respective $\boldsymbol{k}$ values of 0.02, 0.04, 0.06, 0.08 and 0.10 were plotted onto Desmos to compare the effect of increasing the drag force on the motion of the tennis ball

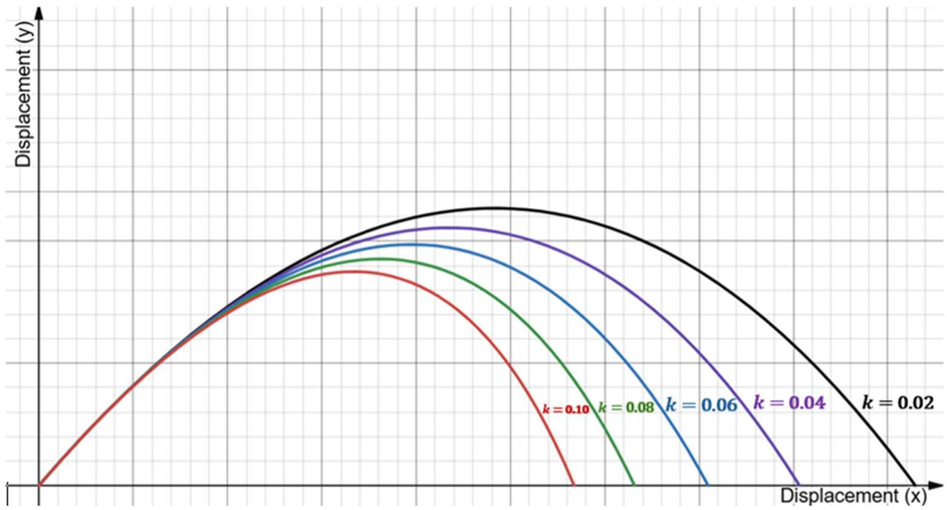


Figure 9: Graphs of $\vec{y}$ as a function of $\vec{x}$ with respective $\mathbf{k}$ values of 0.02, 0.04, 0.06, 0.08 and 0.10

Analysis of the graph:

The black curve, corresponding to a $\boldsymbol{k}$ value of **0.02**, closely represents the ideal parabolic trajectory of a projectile where the only force considered is gravity. This trajectory is nearly symmetric, and the vertex height is close to the maximum height. On the other hand, as k increases, the trajectories deviate significantly from the ideal parabola, becoming increasingly asymmetric and shorter in range and height. The curvature also changes more rapidly and is steeper after the vertex, leading to a sharper curvature as the value of $\boldsymbol{k}$ increases.

Solving for the optimal angles of service ($\boldsymbol{\theta}$ and $\boldsymbol{\phi}$):

For the following calculations solving for the optimal angle of service, a strict assumption that gravity and drag are the only forces acting on the tennis ball must be made for the equations to be accurate. This by extension implies the assumption that there is no energy loss for the tennis ball upon launch or during flight.

First, solving for the angle projected onto the horizontal plane $\boldsymbol{\theta}$ via equations (5) and (6):

$$\frac{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \sin\theta)(1 - e^{-\frac{k}{m}t})}{\left(\frac{m}{k}\right) u\,(\cos\phi \cdot \cos\theta)(1 - e^{-\frac{k}{m}t})} = \frac{\sin\theta}{\cos\theta} = \tan\theta = \frac{(18.28)}{(4.11)}$$

$$\theta = \tan^{-1}\left(\frac{18.28}{4.11}\right) = \mathbf{77.32856676°}$$

Thus, the **optimal angle of service onto the horizontal plane $\boldsymbol{\theta}$** was calculated to be **77.32856676°**.

In contrast to the clear-cut solution for determining $\boldsymbol{\theta}$, solving for $\boldsymbol{\phi}$ on the other hand lacks an elementary (algebraic, exponential, trigonometric) solution. This is especially due to the fact that the terms $\boldsymbol{t}$ and $\mathbf{k}$ both appear linearly in the term $\boldsymbol{k \cdot gt}$, but they also appear exponentially in the term $\boldsymbol{e^{-kt}}$ in the displacement equation for the y axis, which makes isolating and simplifying these terms exponentially impossible (pun-intended). Despite lacking a straightforward solution, an analytical solution can be derived with the help of a **very special function** will come in handy in solving for $\boldsymbol{\phi}$ ...

Application of the Lambert W Function

Although seemingly out of ordinary at first, the Lambert W function $(x = ye^y)$ is precisely the analytic function required in minimizing ϕ in the equation for $y = \left(\frac{m}{k}\right)^2 \left((g + \frac{k}{m} \cdot u \sin\phi) \cdot (1 - e^{-\frac{k}{m}t}) - \frac{k}{m} \cdot gt\right)$.

With the general formula that resolves the linear and exponential terms, we can proceed to solve for t by equating the vertical displacement to -3.00, expanding all the terms and applying the formula.

$$-3.00 = \left(\frac{m}{k}\right)^2 \left((g + \frac{k}{m} \cdot u \sin\phi) \cdot (1 - e^{-\frac{k}{m}t}) - \frac{k}{m} \cdot gt\right)$$

$$\left(\frac{m}{k}\right)^2 \left((g + \frac{k}{m} \cdot u \sin\phi) \cdot (1 - e^{-\frac{k}{m}t}) - \frac{k}{m} \cdot gt\right) + 3.00 = 0$$

Multiplying the equation by $\left(\frac{k}{m}\right)^2$:

$$\left((g + \frac{k}{m} \cdot u \sin\phi) \cdot (1 - e^{-\frac{k}{m}t}) - \frac{k}{m} \cdot gt\right) + 3\left(\frac{k}{m}\right)^2 = 0$$

Expanding and rearranging the terms in the equation into the form $a\omega + b + ce^{d\omega} = 0$:

$$g - ge^{-\frac{k}{m}t} + \frac{k}{m} \cdot u \sin\phi - \frac{k}{m} \cdot u \sin\phi \, e^{-\frac{k}{m}t} - \frac{k}{m} \cdot gt + 3\left(\frac{k}{m}\right)^2 = 0$$

$$-\frac{k}{m} \cdot gt + 3\left(\frac{k}{m}\right)^2 + g + \frac{k}{m} \cdot u \sin\phi - ge^{-\frac{k}{m}t} - \frac{k}{m} \cdot u \sin\phi \, e^{\left(-\frac{k}{m}\right)t} = 0$$

$$-\left(\frac{k}{m} \cdot g\right)t + \left(3\left(\frac{k}{m}\right)^2 + g + \frac{k}{m} \cdot u \sin\phi\right) + \left(-(g + \frac{k}{m} \cdot u \sin\phi)\right)e^{\left(-\frac{k}{m}\right)t} = 0$$

Applying the general formula (1) using the Lambert W function and simplifying the terms:

$$t = -\frac{1}{\left(\frac{k}{m}\right)} W\left(\frac{\left(-\left(g + \frac{k}{m} \cdot u \sin\phi\right)\right)\left(-\frac{k}{m}\right)}{\left(-\left(\frac{k}{m} \cdot g\right)\right)} \cdot e^{-\frac{\left(3\left(\frac{k}{m}\right)^2 + g + \frac{k}{m} u \sin\phi\right)\left(-\frac{k}{m}\right)}{\left(-\left(\frac{k}{m} g\right)\right)}}\right) - \frac{\left(3\left(\frac{k}{m}\right)^2 + g + \frac{k}{m} \cdot u \sin\phi\right)}{\left(-\left(\frac{k}{m} \cdot g\right)\right)}$$

$$t = -\frac{m}{k} W\left(\frac{\left(-\left(g + \frac{k}{m} \cdot u \sin\phi\right)\right)}{g} \cdot e^{-\frac{3\left(\frac{k}{m}\right)^2 + g + \frac{k}{m} u \sin\phi}{g}}\right) + \frac{\left(3\left(\frac{k}{m}\right)^2 + g + \frac{k}{m} \cdot u \sin\phi\right)}{\left(\frac{k}{m} \cdot g\right)}$$

However, in the expression above, there are 3 other unknown variables other than $\boldsymbol{\phi}$ ($\boldsymbol{m}$, $\boldsymbol{k}$ and $\boldsymbol{u}$), hence there is not enough information to uniquely determine all three unknowns. Therefore, fixed values must be set for both $\boldsymbol{m}$, $\boldsymbol{k}$ and $\boldsymbol{u}$, such that differentiation can be carried out with respect to $\boldsymbol{\phi}$ to minimize the value of $\boldsymbol{t}$.

To ensure the proof of concept for the calculations, the **ease of calculation will first be prioritized** by assigning integer coefficients to the terms in the equation; keeping in mind they are to be set to values that do not deviate much from real life values.

Henceforth,

1. The mass of the tennis ball $\boldsymbol{m}$ was set to be **0.056 kg** as the mass of a tennis ball varies from 0.055 kg to 0.0594 kg [7]
2. The constant of proportionality for drag $\boldsymbol{k}$ was set to be **0.056** as the mass of a tennis ball varies from 0.055 kg to 0.065 kg [8]
3. The initial velocity of the tennis ball $\boldsymbol{u}$ was set to be **40m s⁻¹** as the serve speed of a male recreational players (of which demographic I fall under) ranges from 30m s^{-1} to 50m s^{-1} on average [9]

Hence, after substituting these values into the earlier equation, we have time $\boldsymbol{t}$ expressed in terms of $\boldsymbol{\phi}$:

$$t = -\frac{0.056}{0.056} W\left(\frac{\left(-\left(9.0865 + \frac{0.056}{0.056}\cdot 40\sin\phi\right)\right)}{9.0865}\cdot e^{-\frac{3\left(\frac{0.056}{0.056}\right)^2 + 9.8065 + \frac{0.056}{0.056}\cdot 40\sin\phi}{9.8065}}\right) + \frac{\left(3\left(\frac{0.056}{0.056}\right)^2 + 9.8065 + \frac{0.056}{0.056}\cdot 40\sin\phi\right)}{\left(\frac{0.056}{0.056}\cdot 9.8065\right)}$$

$$t = -W\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\right) + \frac{(3 + 9.8065 + 40\sin\phi)}{(9.8065)}$$

Differentiating the expression for $\boldsymbol{t}$ with respect to $\boldsymbol{\phi}$:

$$\frac{dt}{d\phi} = -\frac{d}{d\phi}W\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\right) + \frac{d}{d\phi}\frac{(3 + 9.8065)}{(9.8065)} + \frac{d}{d\phi}\frac{40\sin\phi}{9.8065}$$

$$\frac{dt}{d\phi} = -\frac{d}{d\phi}W\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\right) + \frac{d}{d\phi}\frac{40\sin\phi}{9.8065}$$

Using the derivative of the Lambert W Function $\left(W'(x) = \frac{W(x)}{x\,(1+W(x))}, proof\ in\ appendix\right)$ and applying the chain rule:

$$\frac{dt}{d\phi} = -\frac{W\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin}{9.8065}}\right)\cdot\frac{d}{d\phi}\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40}{9.8065}}\right)}{\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\left(1 + W\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40 si}{9.8065}}\right)\right)} + \frac{40\cos\phi}{9.8065}$$

Applying the product rule for the derivative of $\frac{(-(9.0865+40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}$:

$$\frac{d}{d\phi}\left(\frac{(-(9.0865 + 40\sin\phi))}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\right)$$

$$= \frac{d}{d\phi}\left(-\frac{40\sin\phi}{9.0865} - 1\right)\cdot e^{\left(-\frac{3+9.8065+40\sin\phi}{9.8065}\right)} + \frac{d}{d\phi}\left(e^{-\left(\frac{3+9.8065}{9.8065}+\frac{40\sin\phi}{9.8065}\right)}\right)\cdot\left(-\frac{40\sin\phi}{9.0865} - 1\right)$$

$$= \frac{d}{d\phi}\left(-\frac{40\sin\phi}{9.0865} - 1\right)\cdot e^{\left(-\frac{3+9.8065+40\sin\phi}{9.8065}\right)} + \frac{d}{d\phi}\left(-\left(\frac{3+9.8065}{9.8065}+\frac{40\sin\phi}{9.8065}\right)\right)\cdot\left(e^{-\frac{3+9.8065+40 si}{9.8065}}\right)\cdot\left(-\frac{40\sin\phi}{9.0865} - 1\right)$$

$$= -\frac{40\cos\phi}{9.0865}\cdot e^{-\frac{3+9.8065+40\sin\phi}{9.8065}} - \frac{40\cos\phi}{9.8065}\cdot\left(e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\right)\cdot\left(-\frac{40\sin\phi}{9.0865} - 1\right)$$

Factorize out $e^{-\frac{3+9.8065+40 si}{9.8065}}$:

$$\left(e^{-\frac{3+9.8065+40\sin\phi}{9.8065}}\right)\left(-\frac{40\cos\phi}{9.0865} - \frac{40\cos\phi}{9.8065}\cdot\left(-\frac{40\sin\phi}{9.0865} - 1\right)\right)$$

---

[7] *Tennis ball size & information Guide | Net World Sports*. (n.d.). https://www.networldsports.com/buyers-guides/tennis-balls-guide#:~:text=How%20much%20does%20a%20tennis,(56.0%E2%80%9359.4%20g).
[8] *Mehta, R., Alam, F., & Subic, A. (2008). Review of tennis ball aerodynamics. Sports Technology, 1(1), 7–16. https://doi.org/10.1080/19346182.2008.9648446*
[9] HunterST. (2018, August 8). What is the max potential serve speed for average person? [Online forum post]. Talk Tennis. https://tt.tennis-warehouse.com/index.php?threads/what-is-the-max-potential-serve-speed-for-average-person.572272/page-2

$$= \left(e^{-\frac{3+9.8065\,+\,40\,sin}{9.8065}}\right)\left(-\frac{40\,cos\,\phi}{9.0865} + \frac{40\,cos\,\phi}{9.8065}\cdot\frac{40\,sin\,\phi}{9.0865} + \frac{40\,cos\,\phi}{9.0865}\right)$$

$$= \left(e^{-\frac{3+9.8065\,+\,40\,si}{9.8065}}\right)\left(\frac{40\,cos\,\phi\ \cdot 40\,sin\,\phi}{9.8065}\right)$$

Applying trigonometric identity $2\ sin\,\phi\ \cdot cos\,\phi\ \equiv sin\,2\phi$:

$$\frac{800}{9.8065}\left(e^{-\frac{3+9.8065\,+\,40\,si}{9.8065}}\right)(2\ sin\,\phi\ \cdot cos\,\phi)\ = \frac{800}{9.8065}\left(e^{-\frac{3+9.8065\,+\,40\,si}{9.8065}}\right)(sin\,2\phi)$$

Substitute the expression back into the full derivative:

$$\frac{dt}{d\phi} = -\frac{W\left(\frac{(-(9.0865+40\,sin\,\phi))}{9.0865}\cdot e^{-\frac{3+9.8065\,+\,40\,sin\,\phi}{9.8065}}\right)\cdot\frac{800}{9.8065}\left(e^{-\frac{3+9.8065\,+\,40\,sin\,\phi}{9.8065}}\right)(sin\,2\phi)}{\frac{(-(9.0865+40\,sin\,\phi))}{9.0865}\cdot e^{-\frac{3+9.8065\,+\,40\,si}{9.8065}}\left(1+W\left(\frac{(-(9.0865+40\,sin\,\phi))}{9.0865}\cdot e^{-\frac{3+9.8065\,+\,40\,sin\,\phi}{9.8065}}\right)\right)} + \frac{40\,cos\,\phi}{9.8065}$$

Rearrange the terms and equate the derivative $\frac{dt}{d\phi}$ to 0:

$$\frac{40\,cos\,\phi}{9.8065} - \frac{W\left(\frac{(-(9.0865+40\,sin\,\phi))}{9.0865}\cdot e^{-\frac{3+9.8065\,+\,40\,sin\,\phi}{9.8065}}\right)\cdot\frac{800}{9.8065}\left(e^{-\frac{3+9.8065\,+\,40\,sin}{9.8065}}\right)(sin\,2\phi)}{\frac{(-(9.0865+40\,sin\,\phi))}{9.0865}\cdot e^{-\frac{3+9.8065\,+\,40\,sin}{9.8065}}\left(1+W\left(\frac{(-(9.0865+40\,sin\,\phi))}{9.0865}\cdot e^{-\frac{3+9.8065\,+\,40\,sin}{9.8065}}\right)\right)} = 0$$

Due to the complexity and computational rigour of the equation, I resorted to using Wolfram Alpha (a math engine) to solve it efficiently, as manual calculation was infeasible given the interplay of numerous variables.

Using Wolfram Alpha (code in appendix), $\boldsymbol{\phi}$ was calculated to be $\mathbf{-1.5708°}$.

The result was verified by minimizing the expression for $\boldsymbol{t}$ using Wolfram Alpha again (code in appendix):

$$\min\left\{W\left(-\frac{(9.0865+40\sin(x))\exp\left(-\frac{3+9.8065+40\sin(x)}{9.8065}\right)}{9.0865}\right)+\frac{3+9.8065+40\sin(x)}{9.8065}\right\}\approx$$
$$0.151334 \text{ at } x \approx -1.5708$$

Therefore, the optimal angle between the vertical plane and the horizontal plane $\boldsymbol{\phi}$ for the tennis serve where only gravity and drag is considered is $\mathbf{-1.5708°}$. This implies that the tennis ball should be served at a slight downward angle to minimize the airtime of the tennis ball.

Nonetheless, even with the thorough calculations, tennis serves tend to be more complex in real life, involving various techniques such as applying a spin to the ball. Thus, it raises the question: **how effective are my theoretical equations in modelling the real-life trajectory of the tennis ball?**

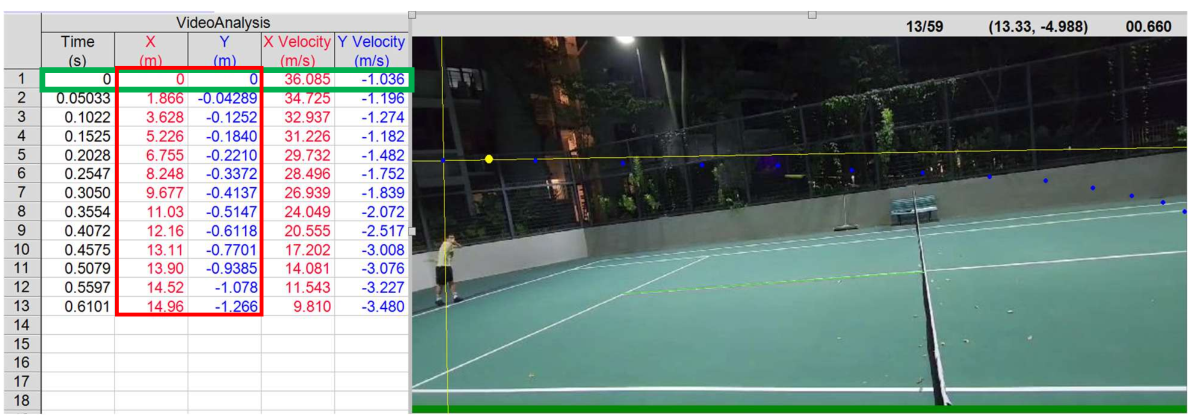
Now, to evaluate the effectiveness of the model and compare it with experimental data, the following methodology was utilized to obtain the experimental trajectory of the tennis ball:

<u>Methodology:</u>

1. Prepare and bring a tennis racket and a tennis ball to the tennis court.
2. Set up two phone cameras with tripods, with one camera set along the height of the tennis court (X axis) and the other camera set along the breadth of the tennis court, as represented below in **figure 10**:

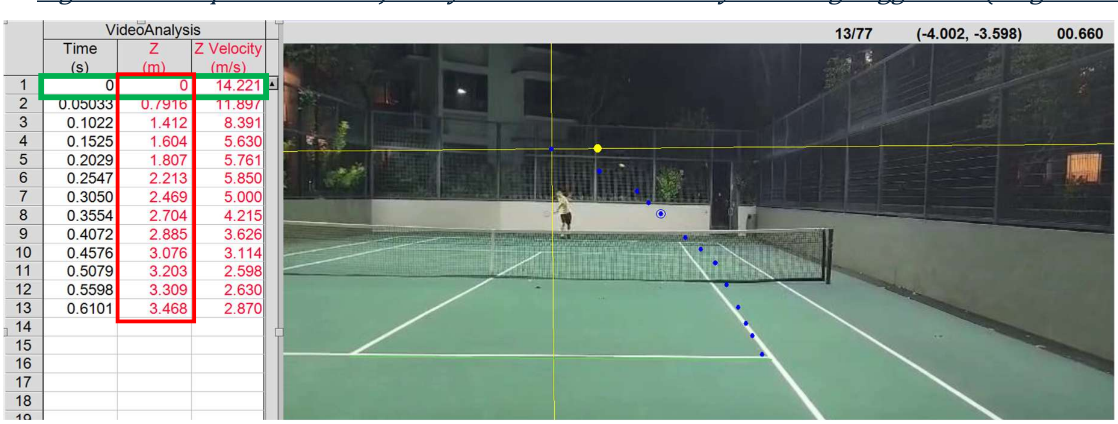
Figure 10: Positions of camera set up on the tennis court

3. Start filming with both cameras by pressing the button to start recording.
4. Serve the tennis ball from the centre of the baseline, represented by the blue dot ● in **figure 10** above.
5. Repeat step 4 until a desirable serve that lands the ball on the right end of the service line (represented by the purple dot ●) on the other side of the court is hit, as depicted in **figure 2.**
6. Review the video footages and trim the videos such that the video of the desirable serve from both perspectives is extracted
7. Utilizing the tracker application **Logger Pro**, upload and analyse the video footage along the height of the court (**Y axis against Z axis**) as well as the video footage along the breadth of the court (**Y axis against Z axis**) frame by frame. At each frame, plot points to obtain the tennis ball's vertical and horizontal displacements and velocities at fixed time intervals, as shown in figures 11.a and 11.b below:



	VideoAnalysis				
	Time (s)	X (m)	Y (m)	X Velocity (m/s)	Y Velocity (m/s)
1	0	0	0	36.085	-1.036
2	0.05033	1.866	-0.04289	34.725	-1.196
3	0.1022	3.628	-0.1252	32.937	-1.274
4	0.1525	5.226	-0.1840	31.226	-1.182
5	0.2028	6.755	-0.2210	29.732	-1.482
6	0.2547	8.248	-0.3372	28.496	-1.752
7	0.3050	9.677	-0.4137	26.939	-1.839
8	0.3554	11.03	-0.5147	24.049	-2.072
9	0.4072	12.16	-0.6118	20.555	-2.517
10	0.4575	13.11	-0.7701	17.202	-3.008
11	0.5079	13.90	-0.9385	14.081	-3.076
12	0.5597	14.52	-1.078	11.543	-3.227
13	0.6101	14.96	-1.266	9.810	-3.480
14					
15					
16					
17					
18					

Figure 11.a: Experimental trajectory of the tennis ball analysed using Logger Pro (Y against X)



	VideoAnalysis		
	Time (s)	Z (m)	Z Velocity (m/s)
1	0	0	14.221
2	0.05033	0.7916	11.897
3	0.1022	1.412	8.391
4	0.1525	1.604	5.630
5	0.2029	1.807	5.761
6	0.2547	2.213	5.850
7	0.3050	2.469	5.000
8	0.3554	2.704	4.215
9	0.4072	2.885	3.626
10	0.4576	3.076	3.114
11	0.5079	3.203	2.598
12	0.5598	3.309	2.630
13	0.6101	3.468	2.870
14			
15			
16			
17			
18			

Figure 11.b: Experimental trajectory of the tennis ball analysed using Logger Pro (Y against Z)

The raw data obtained from the tables will be included in the appendix

8. Plot the values for the displacement in the Y axis against the displacement in the X axis obtained from the Video Analysis table data (Bounded by the rectangle in red) on Desmos in the format of scatter plots. Repeat likewise for the displacement in the Y axis against the displacement in the Z axis, as shown in figure 12 below:

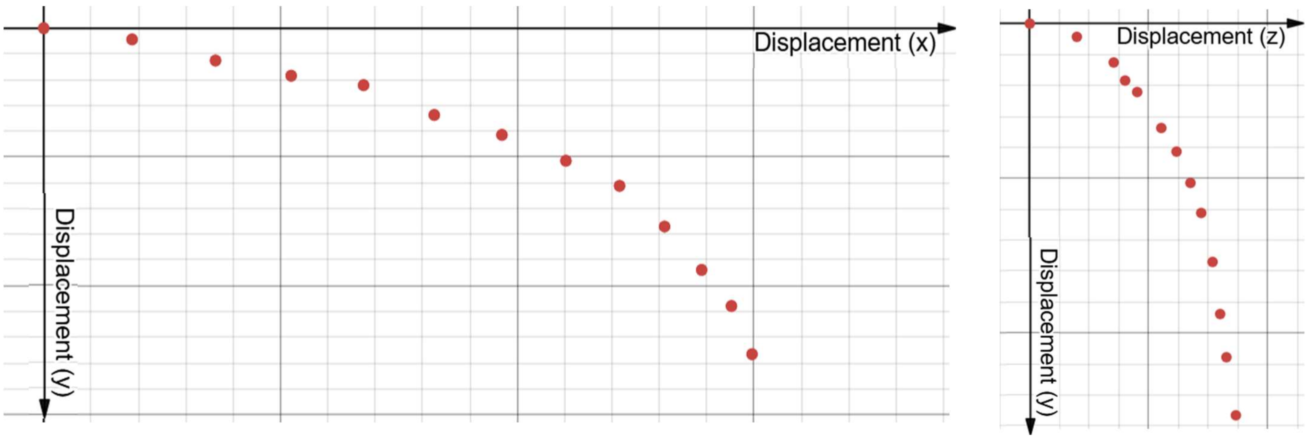


Figure 12: Experimental trajectory of the tennis ball plotted onto Desmos (Left: Y against X, Right: Y against Z)

9. Using the experimental data collected from Desmos (the first two rows of data bounded by the rectangle in green) and other data, calculate the value of the unknown constants for equation (12) and (13):

Times (s)	X displacement (m)	Y displacement (m)	Z displacement (m)	X Velocity (m s^{-1})	Y Velocity (m s^{-1})	Z Velocity (m s^{-1})
0	0	0	0	36.085	-1.035	14.221

Table 1: Table of experimental data collected from Desmos

Referencing equation 12 to identify unknown constants (involving $\boldsymbol{k}, \boldsymbol{u}, \boldsymbol{\phi}, \boldsymbol{\theta}, \boldsymbol{m}$) to be found:

$$\vec{y}(\vec{x}) = \frac{g\left(\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)\vec{x}}{\boldsymbol{u}\,(\boldsymbol{cos\,\phi \cdot sin\,\theta})} + \frac{\boldsymbol{tan\,\phi}}{\boldsymbol{sin\,\theta}}\vec{x} + \left(\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)^2 \cdot g \cdot ln\left|1 + \frac{\vec{x}}{\left(\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)\boldsymbol{u}\,(\boldsymbol{cos\,\phi \cdot sin\,\theta})}\right| \quad --- \quad (12)$$

$$\vec{y}(\vec{z}) = \frac{g\left(\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)\vec{z}}{\boldsymbol{u}\,(\boldsymbol{cos\,\phi \cdot cos\,\theta})} + \frac{\boldsymbol{tan\,\phi}}{\boldsymbol{cos\,\theta}}\vec{z} + \left(\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)^2 \cdot g \cdot ln\left|1 + \frac{\vec{z}}{\left(\frac{\boldsymbol{m}}{\boldsymbol{k}}\right)\boldsymbol{u}\,(\boldsymbol{cos\,\phi \cdot sin\,\theta})}\right| \quad --- \quad (13)$$

a. $\boldsymbol{u}\,(\boldsymbol{cos\,\phi \cdot sin\,\theta})$: This expression is the initial x-component of velocity of the tennis ball from figure 6, it was determined by Logger Pro software to be 36.085m s-1

b. $\boldsymbol{u}\,(\boldsymbol{cos\,\phi \cdot cos\,\theta})$: This expression is the initial z-component of velocity of the tennis ball from figure 6, it was determined by Logger Pro software to be 14.221m s^{-1}

c. $\frac{\boldsymbol{tan\,\phi}}{\boldsymbol{sin\,\theta}}$ and $\frac{\boldsymbol{tan\,\phi}}{\boldsymbol{cos\,\theta}}$: This expression can be calculated by using the dividing vertical initial velocity component u_y by the horizontal initial velocity components u_x and u_z respectively as follows :

$$\frac{u_y}{u_x} = \frac{u\,sin\,\phi}{u\,(cos\,\phi \cdot sin\,\theta)} = \frac{tan\,\phi}{sin\,\theta}\ ;\ \frac{u_y}{u_z} = \frac{u\,sin\,\phi}{u\,(cos\,\phi \cdot cos\,\theta)} = \frac{tan\,\phi}{cos\,\theta}$$

Substituting the initial velocity values obtained from Logger Pro, we obtain:

$$\frac{u_y}{u_x} = \frac{tan\,\phi}{sin\,\theta} = \frac{-1.035}{36.085} = -0.02868227795\ ;\ \frac{u_y}{u_z} = \frac{tan\,\phi}{cos\,\theta} = \frac{-1.035}{14.221} = -0.072779692$$

d. $\boldsymbol{\theta}$: The angle projected onto the horizontal can be calculated as such:

$$\theta = tan^{-1}\left(\frac{sin\ \theta}{cos\ \theta}\right) = tan^{-1}\left(\frac{u\ (cos\ \phi \cdot sin\ \theta)}{u\ (cos\ \phi \cdot cos\ \theta)}\right) = tan^{-1}\left(\frac{u_x}{u_z}\right) = tan^{-1}\left(\frac{36.085}{14.221}\right) = 68.49073564°$$

The experimental angle of $\mathbf{68.49073564°}$ deviates by a moderate amount of about **9°** as compared to the theoretical optimal angle of $\mathbf{77.32856676°}$, indicating that the experimental angle deviates a significant amount from the theoretical angle.

e. $\boldsymbol{\phi}$: The angle between the vertical plane and the horizontal plane can be calculated as such:

$$\phi = tan^{-1}\left(\frac{tan\ \phi}{sin\ \theta} \cdot sin\ \theta\right) = tan^{-1}(-0.02868227795 \cdot sin\ 68.49073564°) = \mathbf{-1.528563393°}$$

The experimental angle of $\mathbf{-1.528563393°}$ deviates by a marginal amount of slightly less than **0.05°** as compared to the theoretical optimal angle of $\mathbf{-1.5708°}$. This indicates that the theoretical angle is highly accurate and consistent with the experimental data.

f. $\boldsymbol{m}$: Using an electronic weighing balance at home, the mass of the tennis ball $\boldsymbol{m}$ was determined to be **0.056 kg**.

g. $\boldsymbol{k}$: The coefficient of proportionality for the air resistance experienced by a tennis ball was derived from a scientific article to be **0.055** [10]

Thus, by substituting these values into equation (12) and (13), the theoretical equations for modelling the trajectory of the tennis ball for the vertical displacement $\vec{\boldsymbol{y}}$ in terms of the horizontal displacements ($\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{z}}$) are:

$$\vec{y}(\vec{x}) = \frac{9.8065\left(\frac{0.056}{0.055}\right)\vec{x}}{36.085} + 0.02868227795\vec{x} + \left(\frac{0.056}{0.055}\right)^2 \cdot 9.8065 \cdot ln\left|1 + \frac{\vec{x}}{\left(\frac{0.056}{0.055}\right)36.085}\right| \quad --- (14)$$

$$\vec{y}(\vec{z}) = \frac{9.8065\left(\frac{0.056}{0.055}\right)\vec{z}}{14.221} + 0.072779692\ \vec{z} + \left(\frac{0.056}{0.055}\right)^2 \cdot 9.8065 \cdot ln\left|1 + \frac{\vec{z}}{\left(\frac{0.056}{0.055}\right)14.221}\right| \quad --- (15)$$

10. Next, plot equations (14) and (15) onto Desmos (with domains $\vec{x} \geq 0$ and $\vec{z} \geq 0$ as the tennis ball is being displaced in the positive direction) to be compared against their respective scatter plot counterparts as shown in figures 13.a and 13.b below:

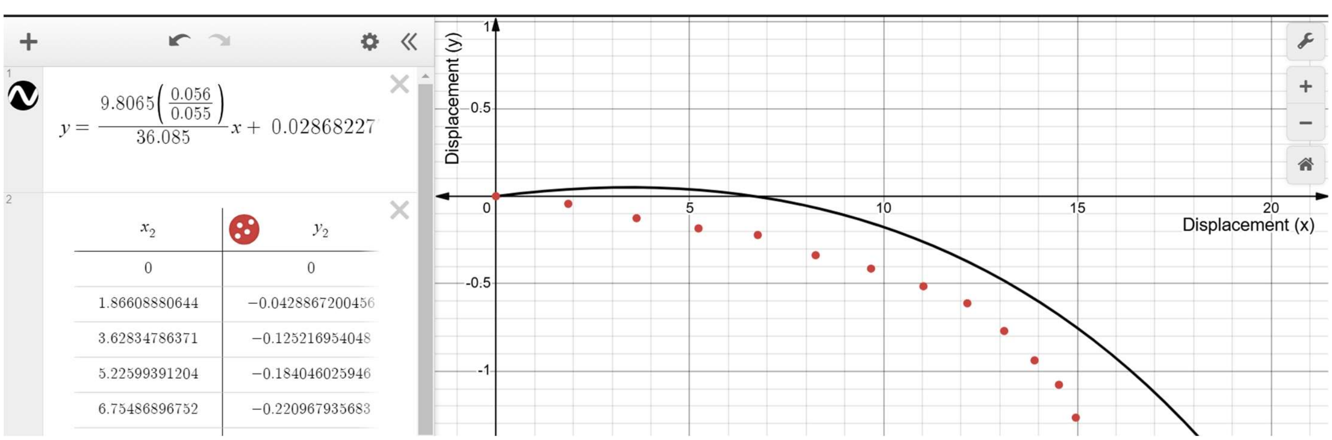


*Figure 13.a: Theoretical trajectory (**in black**) against experimental (in red) (Y against X)*

[10] Chadwick, S.G., & Haake, S.J. (2000). The drag coefficient of tennis balls.

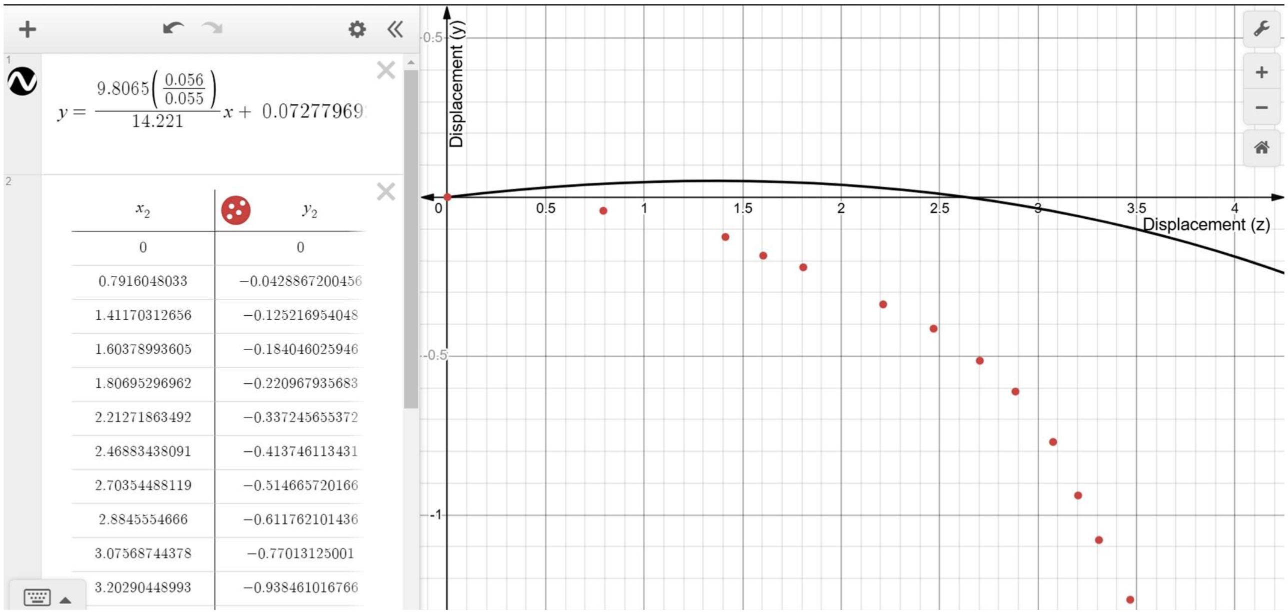


*Figure 13.b: Theoretical trajectory (**in black**) against experimental (in red) (Y against Z)*

11. Next, conduct a statistical analysis to quantify the deviation of the experimental trajectory of the tennis ball by calculating the root mean square error (RMSE) and coefficient of determination (R^2)

   a. Root mean square error (RMSE) formula:

$$\mathrm{RMSE} = \sqrt{\frac{1}{n}\sum_{i=1}^{n}\left(y_{theoretica} - \widehat{y_{experimental}}\right)^2}$$

Where:

- $\boldsymbol{y_{theoretical}}$ is the experimental y displacement value
- $\boldsymbol{\widehat{y_{experimental}}}$ is the theoretical y displacement value
- $\boldsymbol{n}$ is the number of data points

The utility of the RMSE value:

- Magnitude of Error: The RMSE gives a single value that reflects how far off the experimental values are from the experimental values on average. A lower RMSE indicates that the model's predictions are closer to the actual values, meaning the model performs better.
- Units: RMSE is an absolute value, meaning that it is expressed in the same units as the data being modelled, which makes it easy to interpret in context. In this context, since the calculation is for displacement in metres, the RMSE will also be in metres.

   b. Coefficient of determination (R^2) formula:

$$R^2 = 1 - \frac{\sum_{i=1}^{n}\left(y_{theoretical} - \widehat{y_{experimental}}\right)^2}{\sum_{i=1}^{n}\left(y_{theoretical} - \bar{y}\right)^2}$$

Where:

- $\boldsymbol{y_{theoretical}}$ is the experimental y displacement value
- $\boldsymbol{\widehat{y_{experimental}}}$ is the theoretical y displacement value
- $\boldsymbol{n}$ is the number of data points

The utility of the R² value:

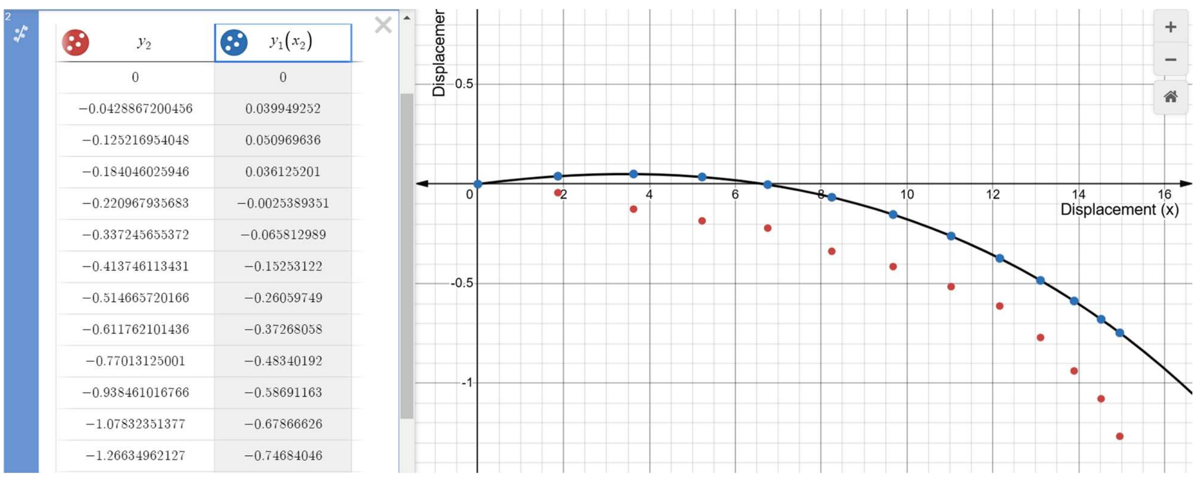
- **Model's Fit**: R^2 gives us a single value that shows how well the model explains the variation in the experimental data. A higher R^2 means the model does a better job of fitting the data.
- **Range**: R^2 is always between 0 and 1.
  - $\mathbf{R^2 = 1}$: The model explains all the variation in the data (perfect fit).
  - $\mathbf{R^2 = 0}$: The model explains none of the variation (poor model).
- **Units**: R^2 is a relative measure and thus a **dimensionless** number. It doesn't have units, which makes it easier to compare models across different types of data.

In short, the **RMSE focuses on how much individual theoretical values deviate from experimental values**, while the $\mathbf{R^2}$ **value measures how well the model explains the variation in the data.**

c. <u>Calculations for the RMSE:</u>

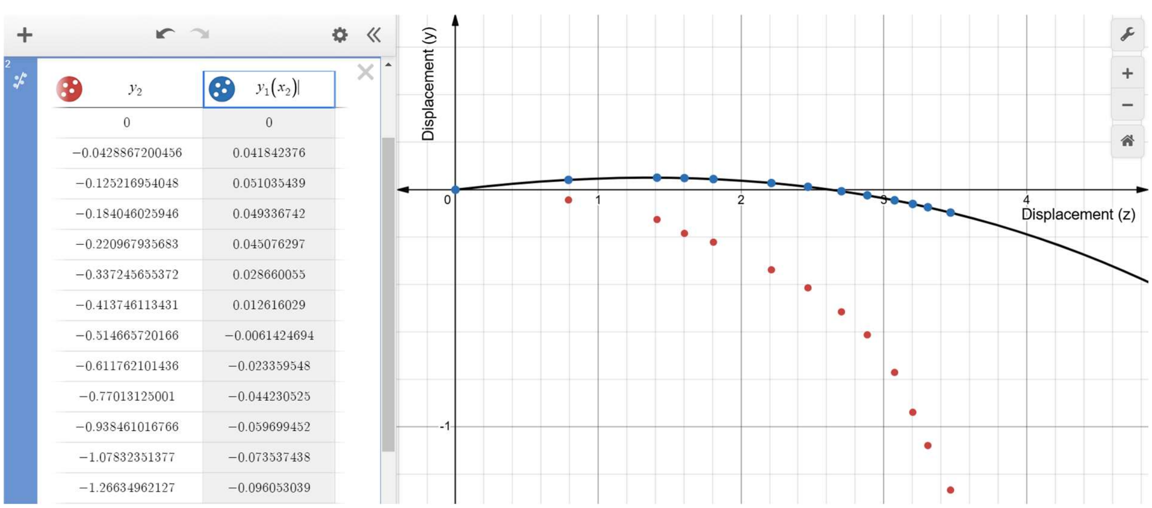
$$RMSE_{\vec{y}(\vec{x})} = \sqrt{\frac{1}{13}\sum_{i=1}^{13}\left(\vec{y}(\vec{x}) - y_2\right)^2}\ ;\ RMSE_{\vec{y}(\vec{z})} = \sqrt{\frac{1}{13}\sum_{i=1}^{13}\left(\vec{y}(\vec{z}) - y_2\right)^2}$$

First, plot a column on the table on Desmos for the functions in the form $\vec{y}(x_2)$ and $\vec{y}(z_2)$ respectively, where x_2 and z_2 are the horizontal displacements obtained from Logger Pro (figures 11.a and 11.b); as shown in figures 14.a and 14.b below:



y_2	$y_1(x_2)$
0	0
−0.0428867200456	0.039949252
−0.125216954048	0.050969636
−0.184046025946	0.036125201
−0.220967935683	−0.0025389351
−0.337245655372	−0.065812989
−0.413746113431	−0.15253122
−0.514665720166	−0.26059749
−0.611762101436	−0.37268058
−0.77013125001	−0.48340192
−0.938461016766	−0.58691163
−1.07832351377	−0.67866626
−1.26634962127	−0.74684046

Figure 14.a: Theoretical trajectory values $\vec{y}(x_2)$ (in blue) with experimental y_2 (in red) (Y against X)



y_2	$y_1(x_2)$
0	0
−0.0428867200456	0.041842376
−0.125216954048	0.051035439
−0.184046025946	0.049336742
−0.220967935683	0.045076297
−0.337245655372	0.028660055
−0.413746113431	0.012616029
−0.514665720166	−0.0061424694
−0.611762101436	−0.023359548
−0.77013125001	−0.044230525
−0.938461016766	−0.059699452
−1.07832351377	−0.073537438
−1.26634962127	−0.096053039

Figure 14.b: Theoretical trajectory values $\vec{y}(z_2)$ (in blue) with experimental y_2 (in red) (Y against Z)

Next, calculate both RSMEs by inputting the expressions of the form $\sqrt{mean(\vec{y}(x_2) - y_2)^2}$ and $\sqrt{mean(\vec{y}(z_2) - y_2)^2}$ respectively, where y_2 is the corresponding value of vertical displacement obtained from the Logger Pro values, as shown in figures 15.a and 15.b below:

$$\sqrt{\text{mean}\left(\left(y_1(x_2) - y_2\right)^2\right)} = 0.281906426961$$

Figure 15.a: RMSE$_{\vec{y}(\vec{x})}$ value

$$\sqrt{\text{mean}\left(\left(y_1(x_2) - y_2\right)^2\right)} = 0.605253856429$$

Figure 15.b: RMSE$_{\vec{z}(\vec{x})}$ value

Therefore, the RMSE values for the theoretical equations of $\vec{\boldsymbol{y}}$ as a function of $\vec{\boldsymbol{x}}$ and $\vec{\boldsymbol{y}}$ as a function of $\vec{\boldsymbol{z}}$ are **0.28190626961m** and **0.605253856429m** respectively, which are indicative of a large magnitude of variance in the data obtained.

This deviation manifests itself in the graph as the $\boldsymbol{y}$ values of the scattered plots in the graph which are the experimental values are noticeably always

lower than that of the theoretical counterpart.

Therefore, the values suggests that the current mode fails to take into consideration other forces or effects that impacts the trajectory of the tennis ball, which calls for improvements and amendments to the model.

d. Calculations for the R² value:

$$R_{\vec{y}(\vec{x})}{}^2 = 1 - \frac{\sum_{i=1}^{n}(\vec{y}(\vec{x}) - y_2)^2}{\sum_{i=1}^{n}(\vec{y}(\vec{x}) - \text{mean}(y_2))^2}\ ;\ R_{\vec{y}(\vec{z})}{}^2 = 1 - \frac{\sum_{i=1}^{n}(\vec{y}(\vec{x}) - y_2)^2}{\sum_{i=1}^{n}(\vec{y}(\vec{x}) - \text{mean}(y_2))^2}$$

Using the previous values from the table plotted in part c. for calculating the RMSE, the expressions to be inputted into Desmos to calculate the R² value are of the form $1 - \frac{\text{total}\left((y_1(x_2) - (y_2))^2\right)}{\text{total}\left((y_1(x_2) - (\text{mean}(y_2)))^2\right)}$ and $1 - \frac{\text{total}\left((y_1(z_2) - (y_2))^2\right)}{\text{total}\left((y_1(z_2) - (\text{mean}(y_2)))^2\right)}$, where the "total" function takes the summation of the $\left(y_1(x_2) - (y_2)\right)^2$ and $\left(y_1(z_2) - (y_2)\right)^2$ expressions.

$$1 - \frac{\text{total}\left(\left(y_1(x_2) - (y_2)\right)^2\right)}{\text{total}\left(\left(y_1(x_2) - \left(\text{mean}(y_2)\right)\right)^2\right)} = 0.446159773395$$

Figure 16.a: R$_{\vec{y}(\vec{x})}{}^2$ value

$$1 - \frac{\text{total}\left(\left(y_1(x_2) - y_2\right)^2\right)}{\text{total}\left(\left(y_1(x_2) - \text{mean}(y_2)\right)^2\right)} = -0.483660437194$$

Figure 16.b: R$_{\vec{y}(\vec{z})}{}^2$ value

Therefore, the R$_{\vec{y}(\vec{x})}{}^2$ value for the theoretical equation for $\vec{\mathbf{y}}$ as a function of $\vec{\boldsymbol{x}}$ is **0.446159773395**, which indicates that theoretical equation is moderately accurate in predicting the experimental data but is far from perfect. On the other hand, the $R_{\vec{y}(\vec{z})}{}^2$ value for the theoretical equation for $\vec{\mathbf{y}}$ as a function of $\vec{\boldsymbol{z}}$ of **−0.483660437194** shows that the theoretical equation fails to model the experimental data and performs worse than the mean.

These values could suggest that the other force or effect that is not considered is much more significant in the theoretical equation for $\vec{y}$ as a function of $\vec{z}$. A possible explanation as to this deviation is that the amount of spin applied to the ball in the $\boldsymbol{z}$ axis is significantly higher than that of the $\boldsymbol{x}$ axis, which results in a greater impact on the acceleration of the tennis ball, resulting in an experimental trajectory that deviates greater than that of the $\boldsymbol{x}$ axis.

Evaluation

The theoretical model showcases significant deviations when compared against the experimental data, which is most probably an indication of the failure of the model to account for the magnus effect (a force that acts on a spinning object moving through air). Upon closer analysis of figures 13.a and 13.b, it is evident that the experimental trajectory of the tennis ball curves down sooner as compared to the theoretical trajectory. This implies that there is a topspin (forward spin applied to the tennis ball along the $\boldsymbol{y}$ axis) which results in a higher $\boldsymbol{y}$ displacement for the same $\boldsymbol{x}$ displacement, creating a steeper downward arc.

The evaluation of the theoretical model against experimental data, however, lacks statistical power and confidence due to a lack of data points. Due to the low frame rate of the phone camera, coupled, with the short duration of the tennis serve (around 0.6 seconds), only 13 frames could be analysed and thus with a small data set it reduces the ability to detect whether deviations from the theoretical trajectory are significant or just due to chance. As such, the limited number of frames makes it harder to confidently say whether the experimental trajectory truly matches the theoretical model.

Nevertheless, the theoretical model showcases it's strengths in that the experimental angles for the initial velocity of the serve $\boldsymbol{\theta}$ and $\boldsymbol{\phi}$ strongly correlate to the theoretical values (in particular the $\boldsymbol{\phi}$ value), which proves that the Lambert W Function is a reliable mathematical tool for projectile motion calculations.

Limitations

1. **Air resistance in reality is not linear to the velocity of the ball**: In reality, air resistance acting on the tennis ball is not strictly linear with velocity, as assumed in the simplified model. For high-speed objects like tennis balls, the relationship between air resistance and velocity transitions between linear $F_{resistance} = -kv$ at lower speeds and quadratic $F_{resistance} = -kv^2$ at higher speeds. This non-linear behaviour arises due to the changing dynamics of drag, particularly as the ball experiences greater turbulent flow at higher velocities. Thus, a more accurate and generalized equation for air resistance is expressed as $F_{resistance} = -kv^n$ where n transitions from 1 (linear) at low velocities to approximately 2 (quadratic) at higher velocities. Failing to consider this factor introduces inaccuracies, particularly under conditions like high-speed serves or spins, where the resistance behaves non-linearly and significantly influences the trajectory.
2. **Lens distortion:** The camera's lens introduces perspective distortion, making it challenging to accurately measure the ball's position. Scaling errors further complicate converting pixel measurements into accurate real-life dimensions.
3. **Frame rate limitation only 30 frames per second:** With only 13 frames available for analysis, the experiment lacks sufficient data points, reducing the statistical reliability and accuracy of the modeled

trajectory. This limited frame rate also increases the potential for missing critical details of the ball's motion.

4. **Other factors that influence the trajectory of the tennis ball are not considered:** The model does not account for external factors like the Magnus effect from spin, the precise force exerted by the racquet, or minor environmental influences such as wind or humidity, leading to discrepancies between experimental and theoretical trajectories.

Extensions

To improve the model in relation to the air resistance calculation, the air resistance-velocity relationship can be further established by determining $F_{resistance} = -kv^n$ for different serving conditions. Another possible extension to be considered for the mathematics exploration is to incorporate spin into the theoretical model in order to increase the accuracy of the theoretical equations. This can be done by using the magnus effect formula wherein the acceleration caused by the spin of the tennis ball is considered. Furthermore, the bounce of the tennis ball through recursion relations can be explored to maximize the velocity of the tennis ball upon its first bounce.
To improve the methodology, professional cameras with a higher frame rate of 60 frames per second can be utilized. Alternatively, a motion tracker can be used on the tennis ball to eliminate the need for frame-by-frame analysis using Logger Pro, which resolves the limitation of lens distortion.

Conclusion

In conclusion, in this mathematics exploration, for a tennis serve with a tennis ball of mass $\boldsymbol{m} = 0.056$ kg, constant of proportionality for drag $\boldsymbol{k} = 0.056$ and initial velocity $\boldsymbol{u} = 40$m s^{-1} where only gravity and drag are considered, the optimal angle of service projected onto the horizontal plane $\boldsymbol{\theta}$ is $\mathbf{77.32856676°}$ and the optimal angle between the vertical plane and the horizontal plane $\boldsymbol{\phi}$ is $-\mathbf{1.5708°}$.

The theoretical model for the trajectory of the tennis ball on the other hand deviates significantly from the experimental trajectory, with RMSE and $\boldsymbol{R^2}$ values for the theoretical equation of $\vec{\boldsymbol{y}}$ as a function of $\vec{\boldsymbol{x}}$ being 0.28190626961m and 0.446159773395 respectively, while that for $\vec{\boldsymbol{y}}$ as a function of $\vec{\boldsymbol{z}}$ being 0.605253856429m and -0.483660437194 respectively.

However, with the incorporation of the magnus effect in my theoretical model, I strongly believe that the theoretical equations will accurately model experimental data and hence will be of great benefit to tennis players in calculating their optimal angle of service.

Bibliography

By. (n.d.). *Tennis Court Dimensions & Size*. ISP. https://www.harrodsport.com/advice-and-guides/tennis-court-dimensions

Edward W. Packel. Edward W. Packel | Lake Forest College. (n.d.). https://www.lakeforest.edu/academics/faculty/edward-w-packel

Kontur-Vid, Sutana, S., Pialhovik, RomanSotola, Bigmouse108, ser_igor, Annetka, Bortonia, Rahadian, V., Yehezkel, Y., MegaShabanov, Yotto, KeithBishop, Axel_wolf, Karelkart, TheArtist, AltoClassic, Tomacco, INDECcraft, … Archive, N. D. (n.d.). *Tennis net stock illustrations*. iStock. https://www.istockphoto.com/illustrations/tennis-net

Ong, N. (2024, May 23). *Tennis kick serve down the T on deuce side - with drills to master it*. Feel Tennis. https://www.feeltennis.net/kick-serve-deuce/

Patrick Mouratoglou. (n.d.). *Tennis serving technique*. Tennis Serving Technique | Mouratoglou Tennis Academy. https://www.mouratoglou.com/en/coaching-corner/tennis-technique/serving-technique/

Tennis ball images - free download on Freepik. Freepik. (n.d.). https://www.freepik.com/free-photos-vectors/tennis-ball

Tomaz. (2024, April 12). *Tennis serve technique: 7 steps to correct serve*. Feel Tennis. https://www.feeltennis.net/serve-technique/

National Institute of Standards and Technology. (n.d.). *Standard gravity* [Value]. National Institute of Standards and Technology. https://physics.nist.gov/cgi-bin/cuu/Value?gn

Chadwick, S.G., & Haake, S.J. (2000). The drag coefficient of tennis balls

HunterST. (2018, August 8). What is the max potential serve speed for average person? [Online forum post]. Talk Tennis. https://tt.tennis-warehouse.com/index.php?threads/what-is-the-max-potential-serve-speed-for-average-person.572272/page-2

Mehta, R., Alam, F., & Subic, A. (2008). Review of tennis ball aerodynamics. Sports Technology, 1(1), 7–16. https://doi.org/10.1080/19346182.2008.9648446

Tennis ball size & information Guide | Net World Sports. (n.d.). https://www.networldsports.com/buyers-guides/tennis-balls-guide#:~:text=How%20much%20does%20a%20tennis,(56.0%E2%80%9359.4%20g).

Appendix

1. Derivative of Lambert W Function:

Since the Lambert W Function can be expressed as $ye^y = x$, we have the following expression:

$$W(x)e^{W(x)} = x$$

Using the product rule on the left-hand side:

$$\frac{d}{dx}\left(W(x)e^{W(x)}\right) = \frac{d}{dx}\ (x)$$

On the left-hand side, apply the product rule:

$$\frac{d}{dx}\left(W(x)e^{W(x)}\right) = \frac{d}{dx}(W(x)) \cdot e^{W(x)} + \frac{d}{dx}\left(e^{W(x)}\right) \cdot W(x)$$

Since $\frac{d}{dx}\left(e^{W(x)}\right) = \frac{d}{dx}(W(x)) \cdot e^{W(x)} = W'(x) \cdot e^{W(x)}$ (by the chain rule), this becomes:

$$\frac{d}{dx}\left(W(x)e^{W(x)}\right) = W'(x) \cdot e^{W(x)} + W'(x) \cdot e^{W(x)} \cdot W(x)$$

On the right-hand side, the derivative of x with respect to x is simply 1, so we now have:

$$W'(x) \cdot e^{W(x)} + W'(x) \cdot e^{W(x)} \cdot W(x) = 1$$

Factor out $W'(x) \cdot e^{W(x)}$

$$\left(W'(x) \cdot e^{W(x)}\right)(1 + W(x)) = 1$$

Rearrange terms to solve for $W'(x)$:

$$W'(x) = \frac{1}{e^{W(x)}(1 + W(x))}$$

Expanding and simplifying the expression using equation 2 $W(x)e^{W(x)} = x$:

$$W'(x) = \frac{1}{e^{W(x)} + W(x) \cdot e^{W(x)}}$$

$$W'(x) = \frac{1}{\frac{x}{W(x)} + x}$$

Multiplying the numerator and denominator by $W(x)$:

$$W'(x) = \frac{W(x)}{x\,(1 + W(x))}$$

2. Two code snippets pasted into wolfram alpha:

```
solve (40*cos(x))/9.8065 - (LambertW[(-((9.0865 + 40*sin(x))/9.8065)) * exp(-(3 + 9.8065 + 40*sin(x))/9.8065)] * (800/9.8065) * exp(-(3 + 9.8065 + 40*sin(x))/9.8065) * sin(2*x)) / ((-((9.0865 + 40*sin(x))/9.8065)) * exp(-(3 + 9.8065 + 40*sin(x))/9.8065) * (1 + LambertW[(-((9.0865 + 40*sin(x))/9.8065)) * exp(-(3 + 9.8065 + 40*sin(x))/9.8065)])) = 0 for x
```

```
LambertW[(-((9.0865 + 40*sin(x))/9.0865)) * exp(-(3 + 9.8065 + 40*sin(x))/9.8065)] + (3 + 9.8065 + 40*sin(x))/9.8065
```

3. Tables of raw data:

Times (s)	X displacement (m)	Y displacement (m)	Z displacement (m)	X Velocity (m s^{-1})	Y Velocity (m s^{-1})	Z Velocity (m s^{-1})
0	0	0	0	36.0849720677	-1.03563342424	14.2212178645
0.0503304524657	1.86608880644	-0.0428867200456	0.7916048033	34.7253856427	-1.19648925753	11.8969678889
0.102186070158	3.62834786371	-0.125216954048	1.41170312656	32.9372976625	-1.27448782793	8.39101474169

0.152516522623	5.22599391204	-0.184046025946	1.60378993605	31.2261969304	-1.18178068769	5.62989233148
0.202846975089	6.75486896752	-0.220967935683	1.80695296962	29.7322446093	-1.48151863797	5.76114685346
0.254702592781	8.24786899556	-0.337245655372	2.21271863492	28.496191467	-1.75241222954	5.850373271
0.305033045247	9.67712240454	-0.413746113431	2.46883438091	26.9389249291	-1.8388962032	4.99967049724
0.355363497712	11.0286204552	-0.514665720166	2.70354488119	24.0487490075	-2.07227938499	4.21483157809
0.407219115404	12.1610860657	-0.611762101436	2.8845554666	20.5552975639	-2.51716924803	3.62586054077
0.45754956787	13.1120475415	-0.77013125001	3.07568744378	17.2018030634	-3.00795510022	3.11375022839
0.507880020336	13.8952887264	-0.938461016766	3.20290448993	14.0808073033	-3.07553302775	2.59754040825
0.559735638027	14.5243685682	-1.07832351377	3.30862292793	11.5425151717	-3.22671678441	2.62996816915
0.610066090493	14.9592849099	-1.26634962127	3.46751003503	9.81000850376	-3.47975257612	2.86967263722