APPLICATION OF GENERATING FUNCTIONS, MODULAR ARITHEMATIC AND ROOTS OF UNITY FOR SUBSET-SUM DIVISIBILITY PROBLEMS

ESSAY OBJECTIVE

By utilizing generating functions, modular arithmetic and roots of unity, how can a general solution to subset-sum divisibility problems be formulated?

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1. Introduction

In 2021, I stumbled across a seemingly innocent question on Reddit asking: “Find the number of all subsets of {1,2,…,2021} such that the sum of the elements in the subset is divisible by 5”.

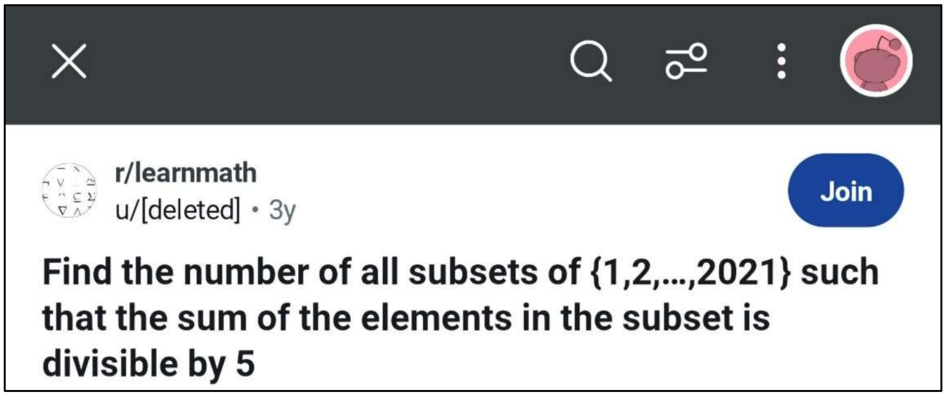

Figure 1: Math question posted on Reddit [1]

Initial attempts at a solution were unsuccessful. Desperate for a solution, I resorted to professional answers on StackExchange.

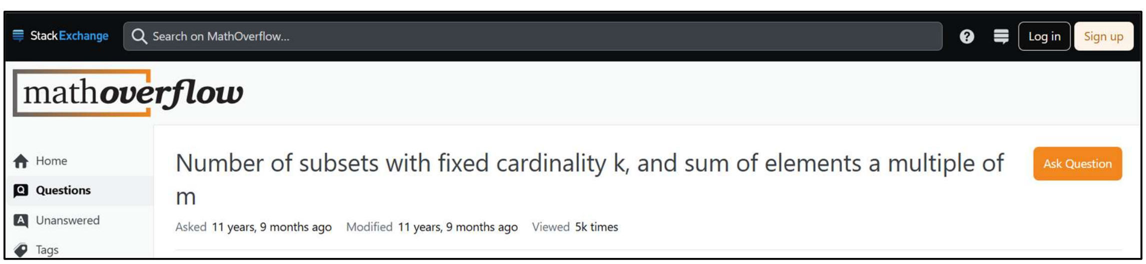

Figure 2: Page discussing answers to a similar form of the question [2]

[1] [deleted]. (2021, December 11). *Find the number of all subsets of {1, 2,…, 2021} such that the sum of the elements in the subset is divisible by 5* [Online forum post]. *Reddit. r/learnmath*. https://www.reddit.com/r/learnmath/comments/rdj27w/find_the_number_of_all_subsets_of_122021_such/

[2] Gaitanas, K. (2013, September 11). *Number of subsets with fixed cardinality k, and sum of elements a multiple of m* [Online forum post]. *MathOverflow*. https://mathoverflow.net/questions/141909/number-of-subsets-with-fixed-cardinality-k-and-sum-of-elements-a-multiple-of-m

I chanced upon a page that discussed a generalized solution for a similar problem for an arbitrary divisor m.

However, the solutions utilized complex university-level math, were computationally intensive throughout and lacked readability.

Serendipitously, by combining fundamental topics such as combinatorics and complex numbers, I realized deriving an intuitive solution was possible.

Therefore, my research question is: "**By utilizing generating functions, modular arithmetic and roots of unity, how can a general solution to subset-sum divisibility problems be formulated?".**

2.The Sub-problem

To identify patterns, we will analyse the smaller set {1,2,3,4}. Its 16 subsets are manageable for detailed inspection but sufficient to reveal generalizable patterns.

Begin by listing out all subsets of {1,2,3,4}:

$\{\}$	$\{4\}$	$\{2,3\}$	$\{1,2,4\}$
$\{1\}$	$\{1,2\}$	$\{2,4\}$	$\{1,3,4\}$
$\{2\}$	$\{1,3\}$	$\{3,4\}$	$\{2,3,4\}$
$\{3\}$	$\{1,4\}$	$\{1,2,3\}$	$\{1,2,3,4\}$

Figure 3: All subsets of {1,2,3,4}

Computing the corresponding sums:

$\{\} \rightarrow 0$	$\{4\} \rightarrow 4$	$\{2,3\} \rightarrow 5$	$\{1,2,4\} \rightarrow 7$
$\{1\} \rightarrow 1$	$\{1,2\} \rightarrow 3$	$\{2,4\} \rightarrow 6$	$\{1,3,4\} \rightarrow 8$
$\{2\} \rightarrow 2$	$\{1,3\} \rightarrow 4$	$\{3,4\} \rightarrow 7$	$\{2,3,4\} \rightarrow 9$
$\{3\} \rightarrow 3$	$\{1,4\} \rightarrow 5$	$\{1,2,3\} \rightarrow 6$	$\{1,2,3,4\} \rightarrow 10$

Figure 4: All subsets of {1,2,3,4} and their corresponding sums

Classify subsets by their sums:

0	**1**	**2**	**3**	**4**
$\{\} \to 0$	$\{1\} \to 1$	$\{2\} \to 2$	$\{3\} \to 3$ $\{1,2\} \to 3$	$\{4\} \to 4$ $\{1,3\} \to 4$
5	**6**	**7**	**8**	**9**
$\{2,3\} \to 5$ $\{1,4\} \to 5$	$\{1,2,3\} \to 6$ $\{2,4\} \to 6$	$\{1,2,4\} \to 7$ $\{3,4\} \to 7$	$\{1,3,4\} \to 8$	$\{2,3,4\} \to 9$
10				
$\{1,2,3,4\} \to 10$				

Figure 6: All subsets classified by their sums from 0 to 10

Classifying subsets by their remainders

0 (mod 5)	**1 (mod 5)**	**2 (mod 5)**	**3 (mod 5)**	**4 (mod 5)**
$\{\} \to 0$ $\{1,4\} \to 5$ $\{2,3\} \to 5$ $\{1,2,3,4\} \to 10$	$\{1\} \to 1$ $\{2,4\} \to 6$ $\{1,2,3\} \to 6$	$\{2\} \to 2$ $\{3,4\} \to 7$ $\{1,2,4\} \to 7$	$\{3\} \to 3$ $\{1,2\} \to 3$ $\{1,3,4\} \to 8$	$\{4\} \to 4$ $\{1,3\} \to 4$ $\{2,3,4\} \to 9$

Figure 5: All subsets classified by their remainders modulo 5

We need a tool that replicates finding the number of subsets with a specific sum **without manual listing**, which would allow classification by remainders modulo 5. Generating functions are precisely the generalizable tool required for this.

2.1 Generating Functions

A generating function is a polynomial product whose terms' coefficients encode the solution to a counting problem when expanded. For the chosen set $\{1,2,3,4\}$, consider the polynomial expansion of $(1+x)(1+x^2)(1+x^3)(1+x^4)$:

$$(1+x)(1+x^2)(1+x^3)(1+x^4)$$

$$= 1 + x + x^2 + x^3 + x^1 \cdot x^2 + x^4 + x^1 \cdot x^3 + x^5 + x^2 \cdot x^3 + x^6 + x^2 \cdot x^4 + x^7 + x^3 \cdot x^4 + x^8 + x^9 + x^{10}$$

$$= 1 + x + x^2 + x^3 + x^3 + x^4 + x^4 + x^5 + x^5 + x^6 + x^6 + x^7 + x^7 + x^8 + x^9 + x^{10}$$

$$= \mathbf{1} + \mathbf{1}x + \mathbf{1}x^2 + \mathbf{2}x^3 + \mathbf{2}x^4 + \mathbf{2}x^5 + \mathbf{2}x^6 + \mathbf{2}x^7 + \mathbf{1}x^8 + \mathbf{1}x^9 + \mathbf{1}x^{10}$$

When compared with the manual classification of subsets by their sums (figure 9), expanding the polynomial acts as an **equivalent algebraic representation** that **generates coefficients** corresponding to the number of subsets.

0	**1**	**2**	**3**	**4**
$\{\} \rightarrow 0$	$\{1\} \rightarrow 1$	$\{2\} \rightarrow 2$	$\{3\} \rightarrow 3$ $\{1,2\} \rightarrow 3$	$\{4\} \rightarrow 4$ $\{1,3\} \rightarrow 4$
5	**6**	**7**	**8**	**9**
$\{2,3\} \rightarrow 5$ $\{1,4\} \rightarrow 5$	$\{1,2,3\} \rightarrow 6$ $\{2,4\} \rightarrow 6$	$\{1,2,4\} \rightarrow 7$ $\{3,4\} \rightarrow 7$	$\{1,3,4\} \rightarrow 8$	$\{2,3,4\} \rightarrow 9$
10				
$\{1,2,3,4\} \rightarrow 10$				

$\boldsymbol{x}$-term	$\boldsymbol{x^0}$	$\boldsymbol{x^1}$	$\boldsymbol{x^2}$	$\boldsymbol{x^3}$	$\boldsymbol{x^4}$
Coefficient	1	1	1	2	2
$\boldsymbol{x}$-term	$\boldsymbol{x^5}$	$\boldsymbol{x^6}$	$\boldsymbol{x^7}$	$\boldsymbol{x^8}$	$\boldsymbol{x^9}$
Coefficient	2	2	2	1	1
$\boldsymbol{x}$-term	$\boldsymbol{x^{10}}$				
Coefficient	1				

Figure 7: A top-bottom comparison of both results from manual computation and polynomial expansion

The generating function works by directly **encoding combinatorial choices into polynomial multiplication**.

In the generating function $(1+x)(1+x^2)(1+x^3)(1+x^4)$, the two terms in each polynomial factor represents two **choices.** For instance, in the factor (x^0+x^3), the x^0 term is the choice to exclude 3 from the subset, while the x^3 term is that to include 3 in the subset. This works because multiplying powers of x adds their exponents, mirroring the summation of elements in a subset.

For instance, choosing to only include 1 and 2 in the subset would be equivalent to:

$$\begin{aligned}
&(1+\boldsymbol{x})(1+\boldsymbol{x^2})(\boldsymbol{1}+x^3)(\boldsymbol{1}+x^4)\\
&= (\boldsymbol{x})(\boldsymbol{x^2})(\boldsymbol{1})(\boldsymbol{1})\\
&= \boldsymbol{x^3}
\end{aligned}$$

This represents the subset $\{1,2\}$ and one way of summing the elements to 3. The only other way to sum the elements to 3 would be choosing to only include 3 in the subset:

$$\begin{aligned}
&(\boldsymbol{1}+\boldsymbol{x})(\boldsymbol{1}+\boldsymbol{x^2})(\boldsymbol{1}+\boldsymbol{x^3})(\boldsymbol{1}+x^4)\\
&= (\boldsymbol{1})(\boldsymbol{1})(\boldsymbol{x^3})(\boldsymbol{1})\\
&= \boldsymbol{x^3}
\end{aligned}$$

When the $\boldsymbol{x^3}$ terms are added together, a coefficient of 2 is obtained. Thus, the **exponents** in the expanded polynomial **represent all possible subset-sums**, while their **coefficients** represent the **number of ways** to form each sum. Generating functions are thus particularly useful as they can be generalized for any number of polynomial factors of any degree.

Extending the polynomial expansion to the original problem, the corresponding generating function for $\{1,2,...,2021\}$ is:

$$f(x) = (1 + x)(1 + x^2) \cdots (1 + x^{2020})(1 + x^{2021})$$

$$= \prod_{k=1}^{2021} (1 + x^k)$$

Expanding the product gives a polynomial of degree 2043231 which is the greatest exponent obtained by multiplying all x terms in each polynomial factor:

$$f(x) = (1 + \boldsymbol{x})(1 + \boldsymbol{x^2}) \cdots (1 + \boldsymbol{x^{2020}})(1 + \boldsymbol{x^{2021}})$$

$$(\boldsymbol{x})(\boldsymbol{x^2}) \cdots (\boldsymbol{x^{2020}})(\boldsymbol{x^{2021}})$$

$$= \boldsymbol{x^{1+2+ \cdots + 2020 + 2021}}$$

Applying the $1 + 2 + \cdots + n = \frac{n(n+1)}{2}$ formula [3]:

$$\boldsymbol{x^{1+2+ \cdots + 2020 + 2021}}$$

$$= \boldsymbol{x^{\frac{2021(2021+1)}{2}}}$$

$$= \boldsymbol{x^{2043231}}$$

This corresponds to **one** possible subset that chooses all elements in {1,2,…2021}, yielding a sum of 2043231.

By assigning arbitrary constants to each term in the expanded form, it can be expressed as such:

$$f(x) = \sum_{n=0}^{2043231} c_n x^n = c_0 + c_1 x + c_2 x^2 + \cdots + c_{2043230} x^{2043230} + c_{2043231} x^{2043231}$$

[3] Chong, The Antipodes, p. 46

The expanded form yields all coefficients and adding up coefficients of x whose exponents are multiples of 5 gives us the answer. However, this expansion is as computationally infeasible as checking every subset.

Hence, we return to the factorized form and utilize strategic substitutions to specifically evaluate for the desired coefficients.

$$\prod_{k=1}^{2021}(1+x^k) = (1+x)(1+x^2)\cdots(1+x^{2020})(1+x^{2021})$$

We seek a number z such that:

For $f(z) + f(z^2) + f(z^3) + f(z^4) + f(z^5)$:

$$\sum c_n x^n = 5x^n, \qquad when\, n \equiv 0\ (mod\ 5)$$

$$\sum c_n x^n = 0, \qquad when\, n \not\equiv 0\ (mod\ 5)$$

Hence, z has to alternate, or rather, **cycle through 5 distinct values**. A visual representation of the exponentiation of z can be visually expressed as a 5-hour clock, albeit representing an **equivalent relationship between successive powers instead of a congruent one**.

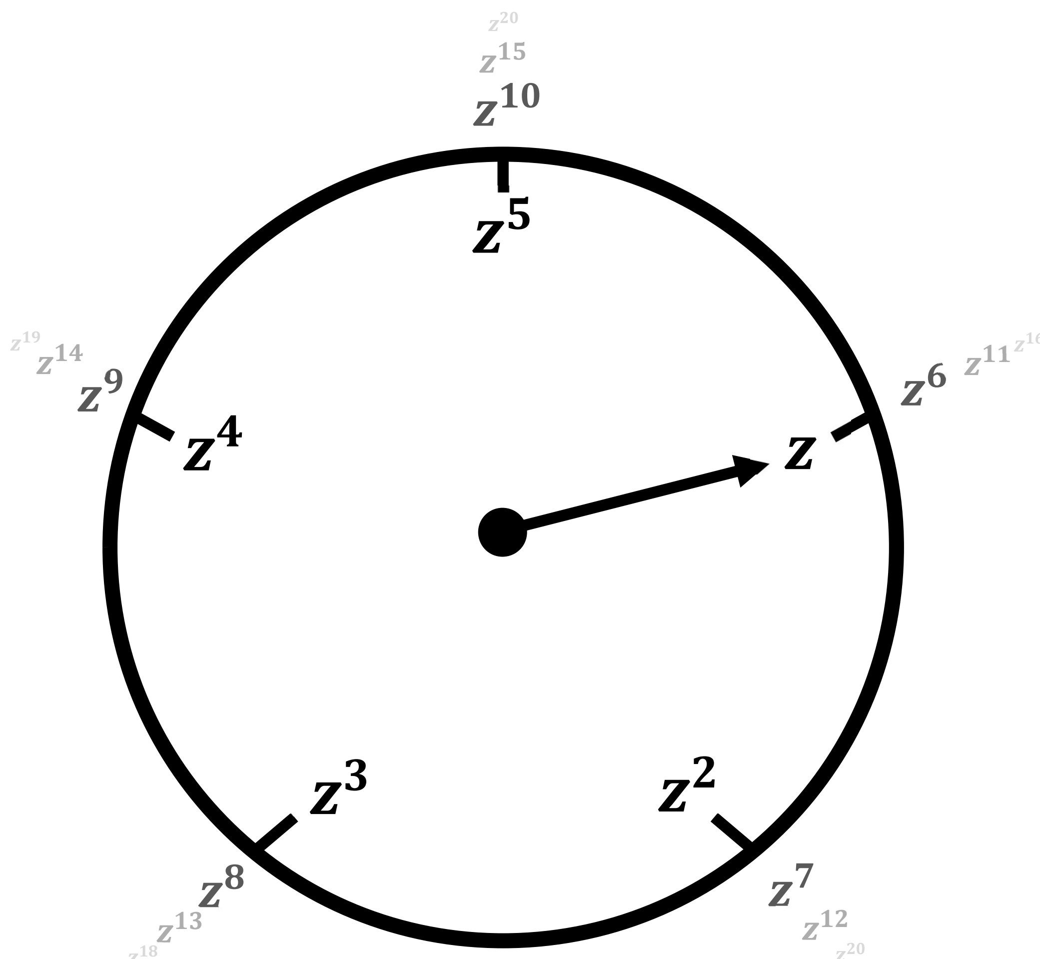

Figure 8: "5-hour" clock that cycles through the values of z, z^2, z^3, z^4 and z^5

Since $z = z^6$:

$$z^6 - z = 0$$

$$z(z^5 - 1) = 0, \qquad since\ z\ \neq 0$$

$$z^5 - 1 = 0$$

To solve for z, we use the technique of roots of unity in the complex plane.

2.2 Roots of unity

For $n \in \mathbb{N}$, the n-th roots of unity [4] are all complex numbers satisfying the equation $w^n - 1 = 0$.

$$w^n - 1 = 0, \qquad w \in \mathbb{C}$$

1 is known as the trivial root of unity, and the principal n-th root of unity is denoted by ζ_n. The rest of the n-th roots are denoted as $\zeta_n, {\zeta_n}^2, {\zeta_n}^3, \cdots, {\zeta_n}^{n-1}$. The integration of complex numbers hence justifies the extension into the complex plane. For simplicity, let the principal 5th root of unity ζ_5 be denoted by z.

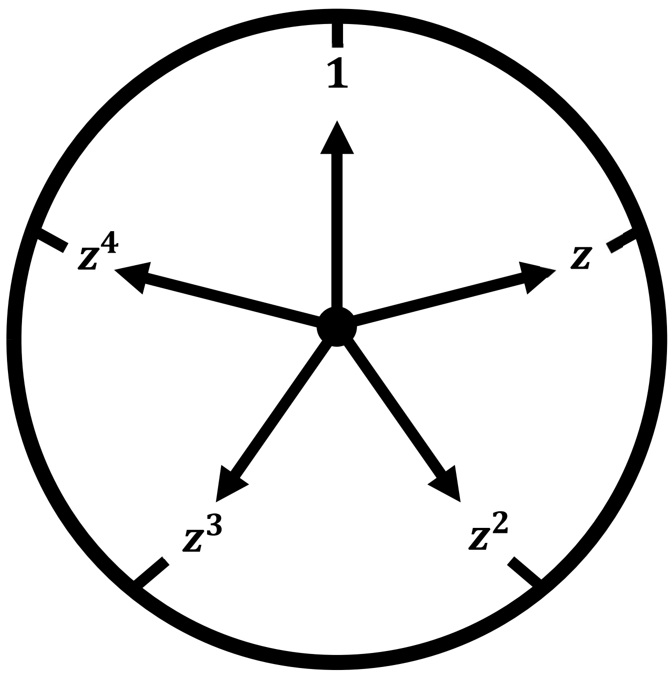

Figure 9: 5th roots of unity 1,z,z²,z³,z⁴ represented by a hour 5-hour clock

To determine z, z^2, z^3 and z^4, the above "cyclic" 5th roots of unity will be mapped onto the complex plane:

[4] Chong, The Antipodes, p. 199

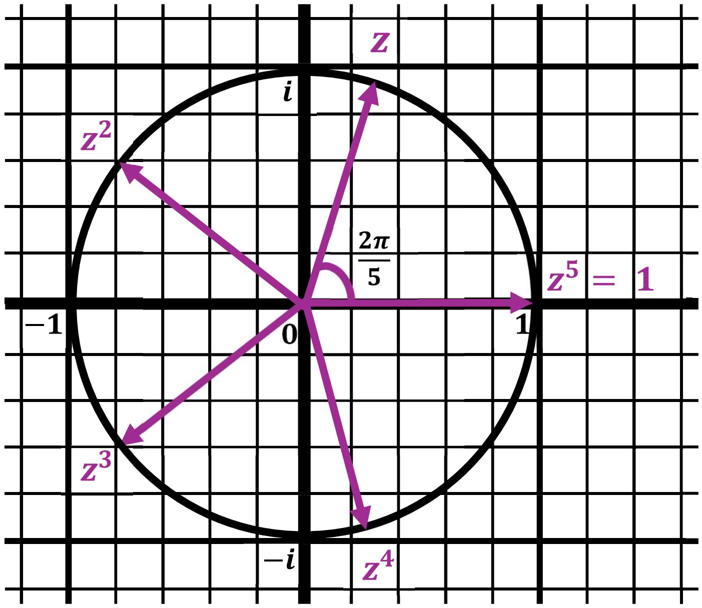

Figure 10: 5th root of unity mapped onto complex plane

By utilizing basic trigonometry, we yield the value of z:

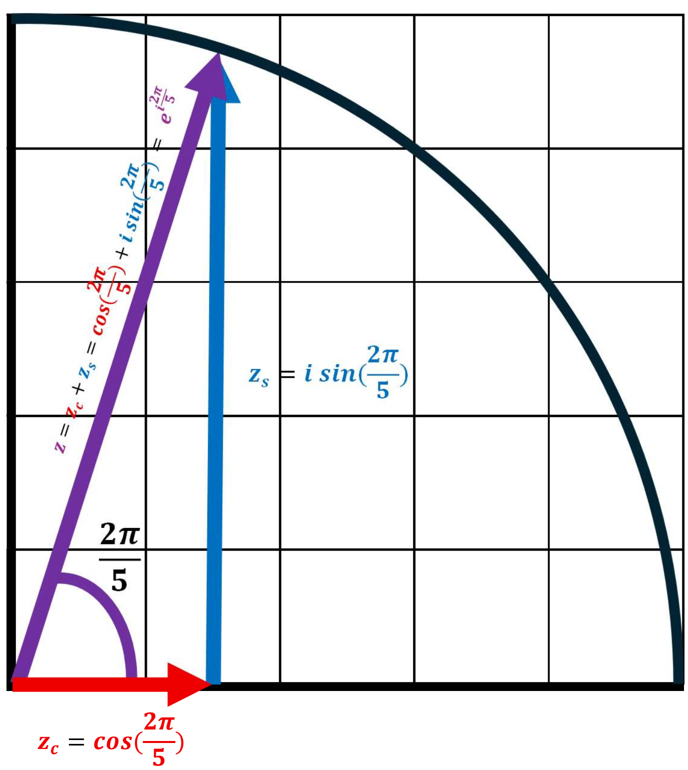

Figure 11: Determine the value of z by inferring its real and imaginary parts via trigonometry

By Euler's formula [5]:

$$e^{i\theta} = cos(\theta) + i\ sin(\theta)$$
$$z = cos(\frac{2\pi}{5}) + i\ sin(\frac{2\pi}{5}) = e^{i\frac{2\pi}{5}}$$

The principal 5th root of unity is hence $e^{i\frac{2\pi}{5}}$.

Next, by De Moivre's theorem [10]:

$$(cos(\theta) + i\ sin(\theta))^n = \ cos(n\theta) + i\ sin(n\theta)$$

$$z^n = \left(cos\left(\frac{2n\pi}{5}\right) + i\ sin\left(\frac{2n\pi}{5}\right)\right)^n = cos\left(\frac{2n\pi}{5}\right) + i\ sin\left(\frac{2n\pi}{5}\right)$$

$$\begin{aligned} z^2 &= cos\left(\frac{4\pi}{5}\right) + i\ sin\left(\frac{4\pi}{5}\right) \\ z^3 &= cos\left(\frac{6\pi}{5}\right) + i\ sin\left(\frac{6\pi}{5}\right) \\ z^4 &= cos\left(\frac{8\pi}{5}\right) + i\ sin\left(\frac{8\pi}{5}\right) \\ z^5 &= cos\left(\frac{10\pi}{5}\right) + i\ sin\left(\frac{10\pi}{5}\right) \\ &= cos(2\pi) + isin(2\pi) \\ &= 1 + i(0) \\ &= 1 \end{aligned}$$

Subsequently, we must verify that $f(z) + \ f(z^2) + f(z^3) + \ f(z^4) + f(z^5)$ isolates coefficients for terms whose exponents are multiples 5.

Expanding each term:

$$\boldsymbol{f(z)} = c_0 + c_1(z) + c_2(z)^2 + c_3(z)^3 + c_4(z)^4 + c_5(z)^5 + \cdots + c_{2043231}(z)^{2043231}$$

$$= c_0 + zc_1 + z^2c_2 + z^3c_3 + z^4c_4 + z^5c_5 + \cdots + c_{2043231}(z)^{2043231}$$

$$= c_0 + zc_1 + z^2c_2 + z^3c_3 + z^4c_4 + c_5 + \cdots + zc_{2043231}$$

$$\boldsymbol{f(z^2)} = c_0 + c_1(z^2) + c_2(z^2)^2 + c_3(z^2)^3 + c_4(z^2)^4 + c_5(z^2)^5 + \cdots + c_{2043231}(z^2)^{2043231}$$

[5] Chong, The Antipodes, p. 198

$$= c_0 + z^2c_1 + z^4c_2 + z^6c_3 + z^8c_4 + z^{10}c_5 + \cdots + z^{4086462}c_{2043231}$$

$$= c_0 + z^2c_1 + z^4c_2 + zc_3 + z^3c_4 + z^5c_5 + \cdots + z^2c_{2043231}$$

$$= c_0 + z^2c_1 + z^4c_2 + zc_3 + z^3c_4 + c_5 + \cdots + z^2c_{2043231}$$

$$\boldsymbol{f}(\boldsymbol{z^3}) = c_0 + c_1(z^3) + c_2(z^3)^2 + c_3(z^3)^3 + c_4(z^3)^4 + c_5(z^3)^5 + \cdots + c_{2043231}(z^3)^{2043231}$$

$$= c_0 + z^3c_1 + z^6c_2 + z^9c_3 + z^{12}c_4 + z^{15}c_5 + \cdots + z^{6129693}c_{2043231}$$

$$= c_0 + z^3c_1 + zc_2 + z^4c_3 + z^2c_4 + z^5c_5 + \cdots + z^3c_{2043231}$$

$$= c_0 + z^3c_1 + zc_2 + z^4c_3 + z^2c_4 + c_5 + \cdots + z^3c_{2043231}$$

$$\boldsymbol{f}(\boldsymbol{z^4}) = c_0 + c_1(z^4) + c_2(z^4)^2 + c_3(z^4)^3 + c_4(z^4)^4 + c_5(z^4)^5 + \cdots + c_{2043231}(z^4)^{2021}$$

$$= c_0 + z^4c_1 + z^8c_2 + z^{12}c_3 + z^{16}c_4 + z^{20}c_5 + \cdots + z^{8172924}c_{2043231}$$

$$= c_0 + z^4c_1 + z^3c_2 + z^2c_3 + zc_4 + z^5c_5 + \cdots + z^4c_{2043231}$$

$$= c_0 + z^4c_1 + z^3c_2 + z^2c_3 + zc_4 + c_5 + \cdots + z^4c_{2043231}$$

$$\boldsymbol{f}\left(\boldsymbol{z^5}\right) = f(1) = c_0 + c_1 + c_2 + c_3 + + c_4 + c_5 \cdots + c_{2043231}$$

Displaying the sum vertically and factorizing the sum:

$$f(z) \;=\; \boldsymbol{c_0} + zc_1 + z^2c_2 + z^3c_3 + z^4c_4 + \boldsymbol{c_5} + zc_6 + z^2c_7 + z^3c_8 + z^4c_9 + \boldsymbol{c_{10}} + \cdots + zc_{2043231}$$

$$f(z^2) = \boldsymbol{c_0} + z^2c_1 + z^4c_2 + zc_3 + z^3c_4 + \boldsymbol{c_5} + z^2c_6 + z^4c_7 + zc_8 + z^3c_9 + \boldsymbol{c_{10}} + \cdots + z^2c_{2043231}$$

$$f(z^3) = \boldsymbol{c_0} + z^3c_1 + zc_2 + z^4c_3 + z^2c_4 + \boldsymbol{c_5} + z^3c_6 + zc_7 + z^4c_8 + z^2c_9 + \boldsymbol{c_{10}} + \cdots + z^3c_{2043231}$$

$$f(z^4) = \boldsymbol{c_0} + z^4c_1 + z^3c_2 + z^2c_3 + zc_4 + \boldsymbol{c_5} + z^4c_6 + z^3c_7 + z^2c_8 + zc_9 + \boldsymbol{c_{10}} + \cdots + z^4c_{2043231}$$

$$f(z^5) = f(1) = \boldsymbol{c_0} + c_1 + c_2 + c_3 + c_4 + \boldsymbol{c_5} + c_6 + c_7 + c_8 + c_9 + \boldsymbol{c_{10}} + \cdots c_{2043231}$$

$$f(z) + f(z^2) + f(z^3) + f(z^4) + f(z^5)$$

$$= 5\boldsymbol{c_0} + (z + z^2 + z^3 + z^4 + 1)c_1 + (z^2 + z + z^4 + z^3 + 1)c_2 + (z^3 + z + z^4 + z^2 + 1)c_3$$

$$+(z^4 + z^3 + z^2 + z + 1)c_4 + 5\boldsymbol{c_5} + (z + z^2 + z^3 + z^4 + 1)c_6 + (z^2 + z + z^4 + z^3 + 1)c_7$$

$$+(z^3 + z + z^4 + z^2 + 1)c_8 + (z^4 + z^3 + z^2 + z + 1)c_9 + 5\boldsymbol{c_{10}} + \cdots + (z + z^2 + z^3 + z^4 + 1)c_{2043231}$$

$$= 5\boldsymbol{c_0} + (1 + z + z^2 + z^3 + z^4)(c_1 + c_2 + c_3 + c_4) + 5\boldsymbol{c_5} + (1 + z + z^2 + z^3 + z^4)(c_6 + c_7 + c_8 + c_9) + 5\boldsymbol{c_{10}} +$$

$$\cdots + 5\boldsymbol{c_{2043230}} + (1 + z + z^2 + z^3 + z^4)c_{2043231}$$

$$= (5\boldsymbol{c_0} + 5\boldsymbol{c_5} + 5\boldsymbol{c_{10}} + \cdots 5\boldsymbol{c_{2043230}}) + (1 + z + z^2 + z^3 + z^4)(c_1 + c_2 + c_3 + c_4 + c_6 + c_7 + c_8 + c_9 + \cdots\ c_{2043231})$$

$$= 5(\boldsymbol{c_0} + \boldsymbol{c_5} + \boldsymbol{c_{10}} + \cdots \boldsymbol{c_{2043230}}) + (1 + z + z^2 + z^3 + z^4)(c_1 + c_2 + c_3 + c_4 + c_6 + c_7 + c_8 + c_9 + \cdots\ c_{2043231})$$

$$= 5\left(\sum c_{5k}\right) + (1 + z + z^2 + z^3 + z^4)(c_1 + c_2 + c_3 + c_4 + c_6 + c_7 + c_8 + c_9 + \cdots\ c_{2043231})$$

Figure 12: Vertical summation of f(z), f(z²), f(z³), f(z⁴) and f(z⁵):

All coefficients for exponents not divisible by 5 $(c_1 + c_2 + c_3 + c_4 + c_6 + c_7 + c_8 + c_9 + \cdots)$ are multiplied by the sum of the 5th roots of unity $(1 + z + z^2 + z^3 + z^4)$. We will next evaluate this sum via geometry.

“Tip-to-tail” vector addition on the complex plane:

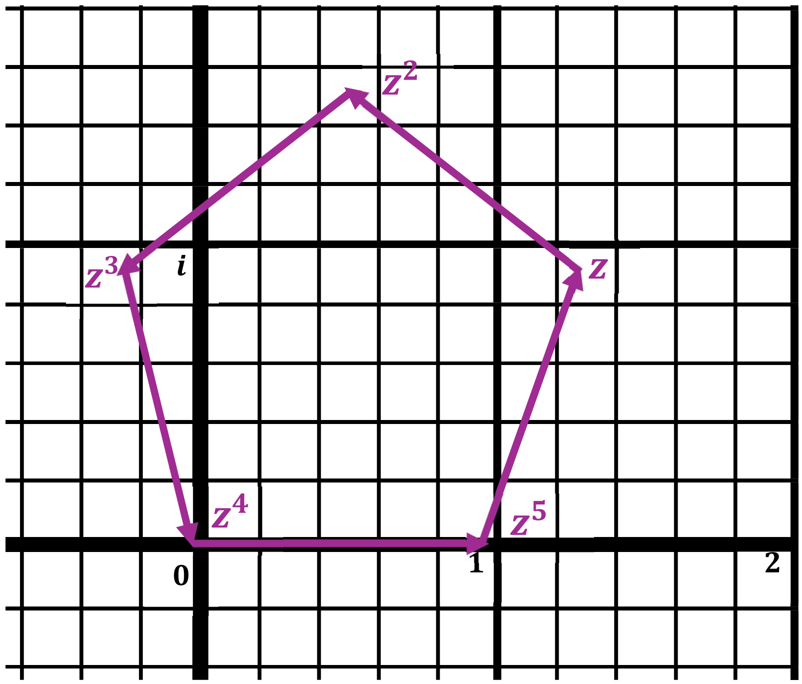

Figure 13: Visual proof for the sum of 5th roots of unity using vector addition on the complex plane

The vectors for the roots $1 + z + z^2 + z^3 + z^4$ form a closed regular pentagon ending at the origin, visually confirming their **sum is zero**.

Alternatively, an algebraic approach produces the same result.

Let $S = 1 + z + z^2 + z^3 + z^4$, and multiply it by z:

$$S = 1 + z + z^2 + z^3 + z^4$$

$$zS = z(1 + z + z^2 + z^3 + z^4) = z + z^2 + z^3 + z^4 + z^5$$

$$= z + z^2 + z^3 + z^4 + 1$$

Equate zS to S:

$$zS = S$$

$$zS - S = 0$$

$$S(1-z) = 0, \qquad since\ z \neq 1$$

$$S = 0$$

$$1 + z + z^2 + z^3 + z^4 = 0$$

Therefore, we have confirmed that:

$$f(z) + f(z^2) + f(z^3) + f(z^4) + f(z^5) = 5 \sum c_{5k}$$

Thus:

$$\sum c_{5k} = \frac{f(z) + f(z^2) + f(z^3) + f(z^4) + f(z^5)}{5}$$

The above expression when evaluated yields the solution. Henceforth, the last step involves computing the generating function at the 5th roots of unity.

2.5 Solution

First, factorize $f(z)$, $f(z^2)$, $f(z^3)$ and $f(z^4)$ into similar forms:

$$f(z) = (1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5) \ \cdots \ (1+z^{2021})$$

Let $g(z) = (1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5)$:

$$f(z) = g(z)(1+z^6)(1+z^7)(1+z^8)(1+z^9)(1+z^{10}) \ \cdots \ (1+z^{2021})$$

$$f(z) = g(z)(1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5) \ \cdots \ (1+z)$$

$$f(z) = g(z)g(z) \ \cdots \ (1+z)$$

To determine the number of times $g(z)$ is repeated, recall the quotient and remainder of 2021 when divided by 5.

$$2021 = 404(5) + 1$$

This means that $g(z)$ contains 404 blocks of the 5 terms $(1+z)(1+z^2)(1+z^3)(1+z^4)(1+z^5)$, each evaluating to $g(z)$, and 1 residual term $(1+z)$.

$$f(z) = \underbrace{g(z)\ \cdots\ g(z)}_{404\text{ times}}(1+z)$$

$$f(z) = (g(z))^{404}\,(1+z)$$

Similarly:

$$f(z^2) = (1+z^2)(1+z^4)(1+z^6)(1+z^8)(1+z^{10})\ \cdots\ (1+z^{4042})$$

$$= (1+z^2)(1+z^4)(1+z)(1+z^3)(1+z^5)\ \cdots\ (1+z^2)$$

$$f(z^2) = (g(z))^{404}\,(1+z^2)$$

$$f(z^3) = (1+z^3)(1+z^6)(1+z^9)(1+z^{12})(1+z^{15})\ \cdots\ (1+z^{6063})$$

$$= (1+z^3)(1+z)(1+z^4)(1+z^2)((1+z^5)\ \cdots\ (1+z^3)$$

$$f(z^3) = (g(z))^{404}\,(1+z^3)$$

$$f(z^4) = (1+z^4)(1+z^8)(1+z^{12})(1+z^{16})(1+z^{20})\ \cdots\ (1+z^{8084})$$

$$= (1+z^4)(1+z^3)(1+z^2)(1+z)(1+z^5)\ \cdots\ (1+z^4)$$

$$f(z^4) = (g(z))^{404}\,(1+z^4)$$

Summing $f(x)$ evaluated at z, z^2, z^3, z^4:

$$f(z) + f(z^2) + f(z^3) + f(z^4)$$

$$= (g(z))^{404}\,(1+z) + (g(z))^{404}\,(1+z^2) + (g(z))^{404}\,(1+z^3) + (g(z))^{404}\,(1+z^4)$$

$$= (g(z))^{404}\,((1+z) + (1+z^2) + (1+z^3) + (1+z^4))$$

$$= (g(z))^{404}\,(4 + z + z^2 + z^3 + z^4)$$

Since $1 + z + z^2 + z^3 + z^4 = 0$:

$$\left(g(z)\right)^{404} (3 + 1 + z + z^2 + z^3 + z^4)$$

$$= (g(z))^{404} (3 + 0)$$

$$= (g(z))^{404} \times 3$$

$$= 3(g(z))^{404}$$

To evaluate $g(z)$, factorize the polynomial $w^5 - 1$ for a strategic substitution:

$$w^5 - 1 = (w - z)(w - z^2)(w - z^3)(w - z^4)(w - z^5)$$

Set $w = -1$:

$$(-1)^5 - 1 = ((-1) - z)((-1) - z^2)((-1) - z^3)((-1) - z^4)((-1) - z^5)$$

$$-1 - 1 = (-1 - z)(-1 - z^2)(-1 - z^3)(-1 - z^4)(-1 - z^5)$$

$$-2 = -(1 + z)(1 + z^2)(1 + z^3)(1 + z^4)(1 + z^5)$$

$$(1 + z)(1 + z^2)(1 + z^3)(1 + z^4)(1 + z^5) = 2$$

Hence:

$$3(g(z))^{404} = 3(2)^{404} = 3(2^{404})$$

Next, $f(z^5)$ is simply $f(1)$, previously evaluated to be 2^{2021}.

Finally, combining the above values yields the exact solution:

$$\sum c_{5k} = \frac{f(z) + f(z^2) + f(z^3) + f(z^4) + f(z^5)}{5} = \frac{\mathbf{2^{2021} + 3(2)^{404}}}{\mathbf{5}}$$

We can verify via modular arithmetic that the solution is an integer.

Expectedly, this value is calculated using Python to be a behemoth number as shown below:

48,156,091,677,115,868,480,078,692,270,323,562,563,127,432,271,414,226,341,441,788,416,392,587,3
32,230,643,768,902,423,100,952,675,139,440,175,832,691,636,710,605,203,448,460,237,564,288,211,0
95,908,952,181,220,994,706,999,213,987,725,600,894,913,657,981,316,441,383,419,013,124,061,043,2
50,886,563,390,130,045,768,759,158,963,219,032,558,271,068,388,678,197,395,169,573,338,427,854,4
89,613,174,086,705,424,669,257,303,162,915,024,788,208,268,264,777,316,890,442,633,681,485,536,7
81,069,346,754,746,178,079,707,116,356,715,945,292,806,889,290,699,278,717,813,583,995,934,722,3
50,764,724,084,592,467,095,871,617,327,975,075,134,165,154,129,579,253,753,628,946,983,886,299,7
83,010,878,803,665,097,455,108,003,028,114,941,219,312,017,966,993,247,554,355,694,609,089,867,8
92,776,900,639,439,367,442,316,226,723,840

Figure 14: Exact value of $(2^{2021}+4(2^{404}))/5$ in numerical form

3. Extensions: Generalized Solutions

3.1 Explicit definition for all n and N

We now seek an explicit formula for the arbitrary set {1,2,3,… N} and divisor n.

Reviewing our specific results:

$$\sum c_{2k} = \frac{f(1) + f(-1)}{2}$$

$$\sum c_{4k} = \frac{f(i) + f(i^2) + f(i^3) + f(i^4)}{4}$$

$$\sum c_{5k} = \frac{f(z) + f(z^2) + f(z^3) + f(z^4) + f(z^5)}{5}, \qquad where\ z = e^{i\frac{2\pi}{5}}$$

It is observed that the number of subset-sums divisible by n is obtained by summing $f(x)$ over the n-th roots of unity and dividing by n.

$$\sum c_{2k} = \frac{f(-1) + f((-1)^2)}{2}, \qquad e^{i\frac{2\pi}{2}} = e^{i\pi} = -1$$

$$\sum c_{4k} = \frac{f(i) + f(i^2) + f(i^3) + f(i^4)}{4}, \qquad e^{i\frac{2\pi}{4}} = e^{i\frac{\pi}{2}} = i$$

Thus, we propose:

$$\sum c_{nk} = \frac{f(\zeta_n) + f\left({\zeta_n}^2\right) + \cdots + f({\zeta_n}^n)}{n}$$

$$= \frac{1}{n}\left(\left(\sum_{\mathrm{i}=1}^{n-1} f\left({\zeta_n}^{\mathrm{i}}\right)\right) + f({\zeta_n}^n)\right)$$

$$= \frac{1}{n}\left(f({\zeta_n}^n) + \left(\sum_{\mathrm{i}=1}^{n-1} f\left({\zeta_n}^{\mathrm{i}}\right)\right)\right)$$

Recall the solution for $f(1)$:

$$f(\zeta_n{}^n) = f(1) = \prod_{k=1}^{N}(1 + 1^k) = \prod_{k=1}^{N}(1 + 1) = \prod_{k=1}^{N}(2) = 2^N$$

Therefore:

$$\sum c_{nk} = \frac{1}{n}\left(2^N + \left(\sum_{\text{i}=1}^{n-1} f\left(\zeta_n{}^{\text{i}}\right)\right)\right), \qquad \zeta_n = e^{i\frac{2\pi}{n}}, n \in \mathbb{N}$$

$$\textit{Where } \zeta_n \textit{ is the principal root of } \; w^n - 1 = 0$$

The formula's validity is contingent on the sum of the n-th roots of unity being 0. As shown with the column addition method, coefficients for exponents not divisible by n are multiplied by this sum, causing them to cancel out and isolate the desired coefficients.

In our specific examples:

<u>$n = 2$</u>:

$$2(\boldsymbol{c_0} + \boldsymbol{c_2} + \boldsymbol{c_4} + \cdots \boldsymbol{c_{2043232}}) + (1 + \alpha)(c_1 + c_3 + \cdots + c_{2043231})$$

$$= 2(\boldsymbol{c_0} + \boldsymbol{c_2} + \boldsymbol{c_4} + \cdots \boldsymbol{c_{2043232}}), \qquad \alpha = -1$$

$$(1 + \alpha)(c_1 + c_3 + \cdots + c_{2043231}) = 0$$

$$(1 + \alpha)\sum c_n \; = 0, \qquad n \not\equiv 0 \; (mod\; 2)$$

<u>$n = 4$</u>:

$$4(\boldsymbol{c_0} + \boldsymbol{c_4} + \boldsymbol{c_{10}} + \cdots \boldsymbol{c_{2043232}}) + (1 + \beta + \beta^2 + \beta^3)(c_1 + c_2 + c_3 + c_5 + c_6 + c_7 + \cdots + c_{2043231})$$

$$= 4(\boldsymbol{c_0} + \boldsymbol{c_4} + \boldsymbol{c_{10}} + \cdots \boldsymbol{c_{2043232}}), \qquad \beta = i$$

$$(1 + \beta + \beta^2 + \beta^3)(c_1 + c_2 + c_3 + c_5 + c_6 + c_7 + \cdots + c_{2043231}) = 0$$

$$(1+\beta+\beta^2+\beta^3)\sum c_n \ = 0, \qquad n \not\equiv 0\ (mod\ 4)$$

$\underline{n=5}$:

$$5(\boldsymbol{c_0}+\boldsymbol{c_5}+\boldsymbol{c_{10}}+\cdots\boldsymbol{c_{2043230}})+(1+z+z^2+z^3+z^4)(c_1+c_2+c_3+c_4+c_6+c_7+c_8+c_9+\cdots c_{2043231})$$

$$=5(\boldsymbol{c_0}+\boldsymbol{c_5}+\boldsymbol{c_{10}}+\cdots\boldsymbol{c_{2043230}}), \qquad z=e^{i\frac{2\pi}{5}}$$

$$(1+z+z^2+z^3+z^4)(c_1+c_2+c_3+c_4+c_6+c_7+c_8+c_9+\cdots c_{2043231})=0$$

$$(1+z+z^2+\cdots+z^4)\sum c_n \ = 0, \qquad n \not\equiv 0\ (mod\ 5)$$

In general:

$$\left(1+\zeta_n+{\zeta_n}^2+{\zeta_n}^3+\cdots+{\zeta_n}^{n-1}\right)\sum c_n \ = 0, \qquad where\ n \not\equiv 0\ (mod\ 5)$$

To prove $1+\zeta_n+{\zeta_n}^2+{\zeta_n}^3+\cdots+{\zeta_n}^{n-1}= \ 0$, observe that $1+\zeta_n+{\zeta_n}^2+{\zeta_n}^3+\cdots$ $+{\zeta_n}^{n-1}$ is a finite geometric series. Hence, the geometric progression formula can be applied [6]:

$$S_n=1+r+r^2+r^3+\cdots+r^{n-1}=\frac{1-r^n}{1-r}$$

$$1+\zeta_n+{\zeta_n}^2+{\zeta_n}^3+\cdots+{\zeta_n}^{n-1}=\frac{1-{\zeta_n}^n}{1-\zeta_n}$$

Recall ζ_n is a root of $w^n-1=0$:

$${\zeta_n}^n-1=0$$

$$\frac{1-\zeta^n}{1-\zeta_n}=\frac{0}{1-\zeta_n}=0, \qquad \zeta\neq 0$$

$$1+\zeta_n+{\zeta_n}^2+{\zeta_n}^3+\cdots+{\zeta_n}^{n-1}=0$$

[6] Chong, The Antipodes, p. 46

Alternatively, by Vieta's formula [7], observe that the coefficient of w^{n-1} in the polynomial $w^n - 1$ is 0, which implies that the sum of roots is 0.

Therefore, we have established the validity of the general formula for all $n \in \mathbb{N}$.

3.2 Closed form solution for varying n and fixed N:

We now generalize the solution for the set {1,2,3,…,2021} and an arbitrary divisor n.

Begin by working out and examining small values of n to uncover patterns:

$$\sum c_{2k} = \frac{2^{2021}}{2}, \qquad \sum c_{3k} = \frac{2^{2021} + 2(2^{673})}{3}, \qquad \sum c_{4k} = \frac{2^{2021}}{4},$$

$$\sum c_{5k} = \frac{2^{2021} + 3(2^{404})}{5}, \qquad \sum c_{6k} = \frac{2^{2021} + 2(2^{673})}{6}$$

$$\sum c_{7k} = \frac{2^{2021} + 3(2^{288})}{7}, \qquad \sum c_{8k} = \frac{2^{2021}}{8}$$

It is observed that for prime divisors 3, 5 and 7, the solution assumes the form:

$$\sum c_{pk} = \frac{2^{2021} + a(2^q)}{p}, \qquad for\ a \in \mathbb{Z}, \qquad p, q \in \mathbb{N}$$

Where p denotes the prime divisor, a is an integer coefficient and q is the quotient of 2021 when divided by p.

Next, considering the case for composite numbers c where:

$$c = {p_1}^{a_1} \cdot {p_2}^{a_2} \cdots {p_\ell}^{a_\ell}$$

$$p_1 \neq p_2 \neq \cdots \neq p_\ell, \qquad all\ primes\ are\ distinct$$

$$a_1, a_2, \cdots, a_\ell\ are\ exponents\ of\ respective\ primes\ p_1, p_2, \cdots, p_\ell$$

---

[7] Chong, The Antipodes, p. 41

A key observation for composite numbers can be made using $n = 6$ (product of primes 2 and 3). While a cursory examination is insufficient to propose a general solution for composite numbers via pattern recognition, the structure of the 6th roots of unity is highly informative. Three of the 6th roots of unity ${\zeta_6}^2, {\zeta_6}^3, {\zeta_6}^4$ are not primitive, instead they are primitive roots of lower orders corresponding to the divisors of 6.

To be explicit, for $n \in \mathbb{N}$, the primitive n-th roots of unity [8] denoted by ${\zeta_n}^i$ are all n-th roots of unity satisfying $\left({\zeta_n}^i\right)^k \neq 1$ for $k \in \{1,2,\ldots,n-1\}$.

${\zeta_6}^2, {\zeta_6}^3, {\zeta_6}^4$ are not primitive 6th roots of unity because they can be raised to a power less than 6 to equal 1, failing the condition $\left({\zeta_n}^i\right)^k \neq 1$.

$$\left({\zeta_6}^3\right)^2 = {\zeta_2}^2 = 1$$
$$\left({\zeta_6}^2\right)^3 = {\zeta_3}^3 = 1$$
$$\left({\zeta_6}^4\right)^2 = \left({\zeta_3}^2\right)^3 = 1$$

Instead ${\zeta_6}^3$ is the 2nd primitive root of unity, while ${\zeta_6}^2, {\zeta_6}^4$ are the primitive 3rd roots of unity.

$${\zeta_6}^3 = \left(e^{i\frac{\pi}{3}}\right)^3 = e^{i\frac{3\pi}{3}} = e^{i\pi} = \zeta_2$$
$${\zeta_6}^2 = \left(e^{i\frac{\pi}{3}}\right)^2 = e^{i\frac{2\pi}{3}} = \zeta_3$$
$${\zeta_6}^4 = \left(e^{i\frac{\pi}{3}}\right)^4 = e^{i\frac{4\pi}{3}} = {\zeta_3}^2$$

[8] Li, R. *Roots of Unity* p. 1

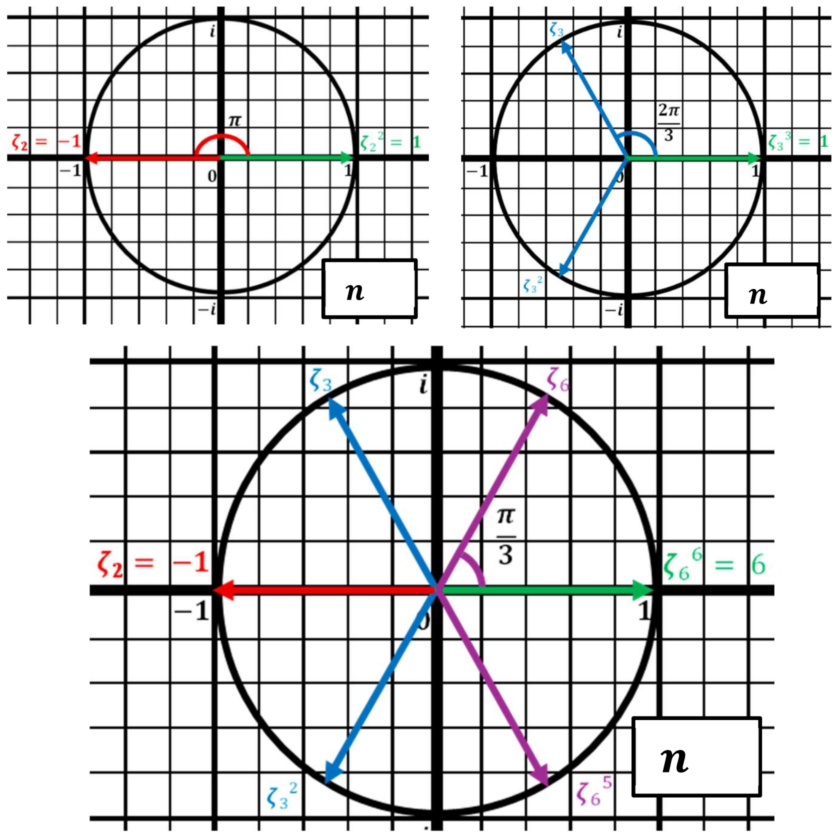

Figure 15: Visual representation of the composite property of the number 6 via the 6th roots of unity's relation to the primitive 2nd and 3rd roots of unity

This shows that for a composite number like 6, the set of its roots includes primitive roots from its divisors (2 and 3), a crucial property for generalizing the solution.

Finally, the solution for divisors 2,4 and 8, which, being powers of 2, are a special case of composite numbers, assumes the form:

$$\sum c_{2^m \cdot k} = \frac{2^{2021}}{2^m}$$

Where 2^m denotes the power of two divisor.

We want to investigate if these proposed forms hold true for all divisors.

3.2.1 Prime divisors

When examining prime divisors p, a key observation is that the set of factors $(1+\zeta_n)(1+{\zeta_n}^2)\cdots(1+{\zeta_n}^n)$ repeats across all $f(\zeta_n), f({\zeta_n}^2), \ldots, f({\zeta_n}^{n-1})$, albeit in different orders. For example, for $n = 7$:

$$f(\zeta_7) = \underline{\boldsymbol{(1+\zeta_7)(1+{\zeta_7}^2)(1+{\zeta_7}^3)(1+{\zeta_7}^4)(1+{\zeta_7}^5)(1+{\zeta_7}^6)}(1+{\zeta_7}^7)}\cdots(1+{\zeta_7}^5)$$

$$f({\zeta_7}^2) = \underline{\boldsymbol{(1+{\zeta_7}^2)(1+{\zeta_7}^4)(1+{\zeta_7}^6)}(1+\zeta_7)\boldsymbol{(1+{\zeta_7}^3)(1+{\zeta_7}^5)}(1+{\zeta_7}^7)}\cdots(1+{\zeta_7}^3)$$

$$f({\zeta_7}^3) = \underline{\boldsymbol{(1+{\zeta_7}^3)(1+{\zeta_7}^6)(1+{\zeta_7}^2)(1+{\zeta_7}^5)(1+\zeta_7)(1+{\zeta_7}^4)}(1+{\zeta_7}^7)}\cdots(1+\zeta_7)$$

$$f({\zeta_7}^4) = \underline{\boldsymbol{(1+{\zeta_7}^4)(1+\zeta_7)(1+{\zeta_7}^5)(1+{\zeta_7}^2)(1+{\zeta_7}^6)(1+{\zeta_7}^3)}(1+{\zeta_7}^7)}\cdots(1+{\zeta_7}^6)$$

$$f({\zeta_7}^5) = \underline{\boldsymbol{(1+{\zeta_7}^5)(1+{\zeta_7}^3)}(1+\zeta_7)\boldsymbol{(1+{\zeta_7}^6)(1+{\zeta_7}^4)(1+{\zeta_7}^2)}(1+{\zeta_7}^7)}\cdots(1+{\zeta_7}^4)$$

$$f({\zeta_7}^6) = \underline{\boldsymbol{(1+{\zeta_7}^6)(1+{\zeta_7}^5)(1+{\zeta_7}^4)(1+{\zeta_7}^3)(1+{\zeta_7}^2)(1+\zeta_7)}(1+{\zeta_7}^7)}\cdots(1+{\zeta_7}^2)$$

The above operation can be visually simplified into a square product table of height and width $n-1$ unit squares. For $n = 7$:

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	8	10	12
3	3	6	9	12	15	18
4	4	8	12	16	20	24
5	5	10	15	20	25	30
6	6	12	18	24	30	36

$\xrightarrow{\textbf{mod } 7}$

$(\textbf{mod } 7)$

	1	**2**	**3**	**4**	**5**	**6**
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

Figure 16: Square product table and Latin square visualization for the distinct ordering of the 7th roots of unity

Employing the above method for any prime p produces a similar Latin square where the **same set of elements are in each row but in different orders**.

Although the above method is helpful as a visual aid, it is **not a definitive proof**, hence **necessitating an algebraic proof** to verifies this property.

An equivalent statement to this property would be:

When $n = p$, for every $A \in \{1,2,\ldots,p-1\}$, A, $2A$, …, $(p-1)A$ are all distinct modulo p.

Assume that there are distinct integers $i, \mathrm{j} \in \{1,2,\ldots,p-1\},\ \ i < \mathrm{j}$ such that there exists a $Ai \equiv A\mathrm{j} \pmod p$, which contradicts the original claim of distinctiveness.

$$Ai \equiv A\mathrm{j} \pmod p$$

$$\mathrm{Ai} - \mathrm{Aj} \equiv 0 \pmod p$$

$$\mathrm{A(i-j)} \equiv 0 \pmod p$$

$$\mathrm{A} \not\equiv 0 (\mathrm{mod}\ p)\ as\ A \in \{1,2,\ldots,p-1\}$$

$$\gcd(\mathrm{A}, p) = 1 \text{ as } \mathrm{A} \in 1,2,\ldots,p-1$$

$$Since\ \mathrm{A} \not\equiv 0(\mathrm{mod}\ p), and\ \gcd(\mathrm{A}, p) = 1\text{ , the condition for division is fulfilled}$$

$$(\mathrm{i}-\mathrm{j}) \equiv 0 \pmod p$$

$$\mathrm{i} \equiv j \pmod p$$

Since $i, \mathrm{j} \in \{1,2,\ldots,p-1\}$, $\mathrm{i} = \mathrm{j}$.

However, **this directly contradicts the original assumption that i$<$ j**.

Therefore, via **proof by contradiction**, the assertion of distinctiveness of A, $2A$, …, $(p-1)A$ modulo p is true.

The crucial step, division by A on both sides of the equation $(\mathrm{i}-\mathrm{j}) \equiv 0 \pmod p$, is **possible only because p is prime**, meaning any $A \in \{1,2,\ldots,p-1\}$ is coprime to p, that is $\gcd(\mathrm{A}, p) = 1$. For example, 1,2,3,…,10 are all coprime to 11. Conversely, this is not true for composite numbers, as a number like 15 has factors 3 and 5 which are not coprime to 15, with $\gcd(3,15) = 3$ and $\gcd(5,15) = 5$. This property translates into the roots of unity $\zeta_p, {\zeta_p}^2, \ldots, {\zeta_p}^{p-1}$ all being primitive when $n = p$. It is due to this property that the **polynomial factors $(\mathbf{1} + \boldsymbol{\zeta_n})(\mathbf{1} + \boldsymbol{\zeta_n}^{\mathbf{2}}) \cdots (\mathbf{1} + \boldsymbol{\zeta_n}^{\boldsymbol{n}})$ will repeat across all $\boldsymbol{f}(\boldsymbol{\zeta_n})$, $\boldsymbol{f}(\boldsymbol{\zeta_n}^{\mathbf{2}})$, …, $\boldsymbol{f}(\boldsymbol{\zeta_n}^{\boldsymbol{n-1}})$**, simplifying the manual calculation.

Henceforth, for all $n = p$ and every $i \in \{1,2, \ldots, p-1\}$,

$$f({\zeta_p}^i) = \left((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{pi})\right)^q \left((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{ri})\right)$$

$$= \left((1 + \zeta_p)(1 + {\zeta_p}^2) \cdots (1 + {\zeta_p}^p)\right)^q \left((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{ri})\right)$$

$$= \left(\left(1 + \zeta_p\right)\left(1 + {\zeta_p}^2\right) \cdots (1 + {\zeta_p}^{(p-1)})(\underline{1 + 1})\right)^q \left(\left(1 + {\zeta_p}^i\right)\left(1 + {\zeta_p}^{2i}\right) \cdots \left(1 + {\zeta_p}^{ri}\right)\right)$$

$$= \left(\underline{2}(1 + \zeta_p)(1 + {\zeta_p}^2) \cdots \left(1 + {\zeta_p}^{(p-1)}\right)\right)^q \left((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{ri})\right),$$

$$Where\ 2021 = q(p) + r.$$

To evaluate $(1 + \zeta_p)(1 + {\zeta_p}^2) \cdots \left(1 + {\zeta_p}^{p-1}\right)$, it is reasonable to generalize it for all p in the form $2m + 1$, as for the only even prime 2, the product trivializes to the single term $1 + \zeta_2$, rendering this case degenerate.

Hence for all $p > 2$ the product of all repeating polynomial factors is:

$$(1 + \zeta_p)(1 + {\zeta_p}^2) \cdots (1 + {\zeta_p}^{(2m+1)})$$

Factoring $w^{2m+1} - 1$:

$$w^{2m+1} - 1 = (w - \zeta_{2m+1})(w - {\zeta_{2m+1}}^2) \cdots (w - {\zeta_{2m+1}}^{2m})(w - {\zeta_{2m+1}}^{2m+1})$$

$$w^{2m+1} - 1 = (w - \zeta_{2m+1})(w - {\zeta_{2m+1}}^2) \cdots (w - {\zeta_{2m+1}}^{2m})(w - 1)$$

$$\frac{w^{2m+1} - 1}{w - 1} = (w - \zeta_{2m+1})(w - {\zeta_{2m+1}}^2) \cdots (w - {\zeta_{2m+1}}^{2m})$$

Set $w = -1$:

$$\frac{(-1)^{2m+1} - 1}{(-1) - 1} = (-1 - \zeta_{2m+1})\left(-1 - {\zeta_{2m+1}}^2\right) \cdots \left(-1 - {\zeta_{2m+1}}^{2m}\right)$$

$$\frac{-1 - 1}{(-1) - 1} = (1 + \zeta_{2m+1})\left(1 + {\zeta_{2m+1}}^2\right) \cdots \left(1 + {\zeta_{2m+1}}^{2m}\right)$$

$$1 = (1 + \zeta_{2m+1})\left(1 + {\zeta_{2m+1}}^2\right) \cdots \left(1 + {\zeta_{2m+1}}^{2m}\right)$$

$$1 = (1 + \zeta_p)(1 + {\zeta_p}^2) \cdots \left(1 + {\zeta_p}^{p-1}\right), \qquad p\ is\ odd$$

Therefore:

$$f\left({\zeta_p}^i\right) = \left(2(1 + \zeta_p)(1 + {\zeta_p}^2) \cdots \left(1 + {\zeta_p}^{p-1}\right)\right)^q ((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{ri}))$$

$$= (2(1))^q((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{ri}))$$

$$= 2^q((1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{ri}))$$

$$= 2^q\left(\prod_{j=1}^{r}\left(1 + {\zeta_p}^{ij}\right)\right) - - - (3)$$

Thus, for all prime divisors p, the general solution is simplified as such:

$$\sum c_{pk} = \frac{2^{2021} + \sum_{i=1}^{p-1} f\left({\zeta_p}^i\right)}{p}$$

$$\sum_{i=1}^{p-1} f\left({\zeta_p}^i\right) = \sum_{i=1}^{p-1} 2^q\left(\prod_{j=1}^{r}\left(1 + {\zeta_p}^{ij}\right)\right) = 2^q \sum_{i=1}^{p-1}\left(\prod_{j=1}^{r}\left(1 + {\zeta_p}^{ij}\right)\right)$$

$$\sum c_{pk} = \frac{2^{2021} + 2^q \sum_{i=1}^{p-1}\left(\prod_{j=1}^{r}\left(1 + {\zeta_p}^{ij}\right)\right)}{p}$$

In spite of the complicated formula, there are simple closed form solutions for $r = 1$, $r = p - 1$ and $r = p - 2$.

For $r = 1$:

$$\sum c_{pk} = \frac{2^{2021} + 2^q \sum_{i=1}^{p-1}\left(1 + {\zeta_p}^i\right)}{p}$$

Computing the summation term:

$$\sum_{i=1}^{p-1}\left(1 + {\zeta_p}^i\right)$$

$$= \sum_{i=1}^{p-1} 1 + \sum_{i=1}^{p-1} {\zeta_p}^i$$

$$= p - 1 + \sum_{i=1}^{p-1} {\zeta_p}^i$$

$$= p - 1 + \left(\zeta_p + {\zeta_p}^2 + \cdots + {\zeta_p}^{p-1}\right), \qquad 1 + \zeta_p + {\zeta_p}^2 + \cdots + {\zeta_p}^{p-1} = 0$$

$$= p - 2 + \left(\underline{1 + \zeta_p + {\zeta_p}^2 + \cdots + {\zeta_p}^{p-1}}\right)$$

$$= p - 2 + (0)$$

$$= p - 2$$

Hence:

$$\sum c_{pk} = \frac{2^{2021} + (p-2)2^q}{p}$$

For $r = p - 1$:

$$\sum c_{pk} = \frac{2^{2021} + 2^q \sum_{i=1}^{p-1}\left(\prod_{j=1}^{p-1}(1 + {\zeta_p}^{ij})\right)}{p}$$

Utilizing the previously established value of the product $(1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{(p-1)i})$:

$$\sum_{i=1}^{p-1}\left(\prod_{j=1}^{p-1}(1 + {\zeta_p}^{ij})\right) = (1 + {\zeta_p}^i)(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{(p-1)i}) = 1$$

Computing the summation term:

$$\sum c_{pk} = \frac{2^{2021} + 2^q \sum_{i=1}^{p-1} 1}{p}$$

$$= \frac{2^{2021} + (p-1)2^q}{p}$$

For $r = p - 2$:

$$\sum c_{pk} = \frac{2^{2021} + 2^q \sum_{i=1}^{p-1}\left(\prod_{j=1}^{p-2}(1 + {\zeta_p}^{ij})\right)}{p}$$

Utilizing the previously established value of the product $(1 + {\zeta_p}^{i})(1 + {\zeta_p}^{2i}) \cdots$ $(1 + {\zeta_p}^{(p-1)i})$:

$$\prod_{j}^{p-2}(1 + {\zeta_p}^{ij}) = (1 + {\zeta_p}^{i})(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{(p-2)i})$$

$$(1 + {\zeta_p}^{i})(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{(p-2)i})(1 + {\zeta_p}^{(p-1)i}) = 1$$

$$(1 + {\zeta_p}^{i})(1 + {\zeta_p}^{2i}) \cdots (1 + {\zeta_p}^{(p-2)i}) = \frac{1}{1 + {\zeta_p}^{(p-1)i}}$$

Computing the summation term for the reciprocal polynomial factor:

$$\sum_{i=1}^{p-1}\left(\frac{1}{1 + {\zeta_p}^{(p-1)i}}\right) = \frac{1}{1 + {\zeta_p}^{(p-1)}} + \frac{1}{1 + {\zeta_p}^{2(p-1)}} + \cdots + \frac{1}{1 + {\zeta_p}^{(p-1)(p-1)}}$$

Utilizing the previous result where A, $2A$, …, $(p-1)A$ are distinct modulo p:

$$\frac{1}{1 + {\zeta_p}^{(p-1)}} + \frac{1}{1 + {\zeta_p}^{2(p-1)}} + \cdots + \frac{1}{1 + {\zeta_p}^{(p-1)(p-1)}}$$

$$= \frac{1}{1 + \zeta_p} + \frac{1}{1 + {\zeta_p}^{2}} + \cdots + \frac{1}{1 + {\zeta_p}^{(p-1)}}$$

Pairing every root by their conjugate pairs:

$$\left(\frac{1}{1+\zeta_p}+\frac{1}{1+\zeta_p{}^{(p-1)}}\right)+\left(\frac{1}{1+\zeta_p{}^2}+\frac{1}{1+\zeta_p{}^{(p-2)}}\right)+\cdots+\left(\frac{1}{1+\zeta_p{}^{\left(\frac{p+1}{2}\right)}}+\frac{1}{1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}}\right)$$

$$=\frac{1+\zeta_p+1+\zeta_p{}^{(p-1)}}{(1+\zeta_p)(1+\zeta_p{}^{(p-1)})}+\frac{1+\zeta_p{}^2+1+\zeta_p{}^{(p-2)}}{(1+\zeta_p{}^2)(1+\zeta_p{}^{(p-2)})}+\cdots+\frac{1+\zeta_p{}^{\left(\frac{p+1}{2}\right)}+1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}}{\left(1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}\right)\left(1+\zeta_p{}^{\left(\frac{p+1}{2}\right)}\right)}$$

$$=\frac{1+\zeta_p+1+\zeta_p{}^{(p-1)}}{1+\zeta_p+\zeta_p{}^p+\zeta_p{}^{(p-1)})}+\frac{1+\zeta_p{}^2+1+\zeta_p{}^{(p-2)}}{1+\zeta_p{}^2+\zeta_p{}^p+\zeta_p{}^{(p-2)}}+\cdots+\frac{1+\zeta_p{}^{\left(\frac{p+1}{2}\right)}+1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}}{1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}+\zeta_p{}^p+\zeta_p{}^{\left(\frac{p+1}{2}\right)}}$$

$$=\frac{1+\zeta_p+1+\zeta_p{}^{(p-1)}}{1+\zeta_p+1+\zeta_p{}^{(p-1)})}+\frac{1+\zeta_p{}^2+1+\zeta_p{}^{(p-2)}}{1+\zeta_p{}^2+1+\zeta_p{}^{(p-2)}}+\cdots+\frac{1+\zeta_p{}^{\left(\frac{p+1}{2}\right)}+1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}}{1+\zeta_p{}^{\left(\frac{p-1}{2}\right)}+1+\zeta_p{}^{\left(\frac{p+1}{2}\right)}}$$

$$=1+1+\cdots+1$$

$$=\sum_{i=1}^{\frac{p-1}{2}}1$$

$$=\frac{p-1}{2}$$

Therefore:

$$\sum c_{pk}=\frac{2^{2021}+\left(\frac{p-1}{2}\right)2^q}{p}$$

Hence, the general solution for the subset-set divisibility problem for prime divisors p is:

$$\sum c_{pk}=\begin{cases}\dfrac{2^{2021}+(p-2)2^q}{p} & r=1\\[2ex] \dfrac{2^{2021}+\left(\frac{p-1}{2}\right)2^q}{p} & r=p-2\\[2ex] \dfrac{2^{2021}+(p-1)2^q}{p} & r=p-1\\[2ex] \dfrac{2^{2021}+2^q\sum_{i=1}^{p-1}\left(\prod_{j=1}^{r}\left(1+\zeta_p{}^{ij}\right)\right)}{p} & otherwise\end{cases}$$

For $r = 1$, we can apply the above solution to the following primes:

$$\boldsymbol{p = 2}$$

$$2021 = 1010(2) + 1$$

$$\frac{2^{2021} + (2-2)2^{1010}}{2} = \frac{2^{2021}}{2}$$

$$\boldsymbol{p = 5}$$

$$2021 = 404(5) + 1$$

$$\frac{2^{2021} + (5-2)2^{404}}{5} = \frac{2^{2021} + 3(2^{404})}{5}$$

$$\boldsymbol{p = 101}$$

$$2021 = 20(101) + 1$$

$$\frac{2^{2021} + (101-2)2^{20}}{5} = \frac{2^{2021} + 99(2^{20})}{5}$$

Next, for $r = p - 2$:

$$\boldsymbol{p = 7}$$

$$2021 = 288(7) + 5 = 288(7) + (7 - 5)$$

$$\frac{2^{2021} + \left(\frac{7-1}{2}\right)2^{288}}{7} = \frac{2^{2021} + 3(2^{288})}{7}$$

$$\boldsymbol{p = 17}$$

$$2021 = 118(17) + 15 = 118(17) + (17 - 15)$$

$$\frac{2^{2021} + \left(\frac{17-1}{2}\right)2^{288}}{17} = \frac{2^{2021} + 8(2^{288})}{17}$$

Next, for $r = p - 1$:

$$\boldsymbol{p = 3}$$

$$2021 = 673(3) + 2 = 673(3) + (3 - 1)$$

$$\frac{2^{2021} + (3-1)2^{673}}{3} = \frac{2^{2021} + 2(2^{673})}{3}$$

$$\boldsymbol{p = 337}$$

$$2021 = 5(337) + 336 = 5(337) + (337 - 1)$$

$$\frac{2^{2021} + (337-1)2^{288}}{337} = \frac{2^{2021} + 336(2^{288})}{337}$$

The results for $p = 2$, $p = 3$, $p = 5$ and $p = 7$ match the earlier solution, **affirming the validity of the general formula** for the specific remainder cases.

3.2.2 Composite divisors

3.2.2.1 Powers of 2

Recalling the proposed solution for divisors that are powers of 2:

$$\sum c_{2^m \cdot k} = \frac{2^{2021}}{2^m}$$

The implication of the above solution is that for $n = 2^m$, all $\sum_{\mathrm{i}=1}^{n-1} f\left(\zeta_n{}^{\mathrm{i}}\right)$ will evaluate to 0. For $n = 2$, $n = 4$ and $n = 8$, we observe that **each $\boldsymbol{f(\zeta_n)}$, $\boldsymbol{f\left(\zeta_n{}^2\right)}$, …, $\boldsymbol{f\left(\zeta_n{}^{n-1}\right)}$ evaluates to 0**. We want to prove that this observation holds true for all 2^m where $m \in \mathbb{N}$.

First, we must solve for the exponent i of ζ_n for which the polynomial factor of $\left(1 + \zeta_n{}^{\mathrm{i}}\right)$ is equal to 0:

$$1 + \zeta_n{}^{\mathrm{i}} = 0$$

$$\zeta_n{}^{\mathrm{i}} = -1$$

$$\left(e^{\left(\frac{i2\pi}{n}\right)}\right)^{i} = e^{i\pi}$$

$$e^{i\left(\frac{i2\pi}{n}\right)} = e^{i\pi}$$

$$i\left(\frac{i2\pi}{n}\right) = i\pi$$

$$i\left(\frac{2}{n}\right) = 1$$

$$i = \frac{n}{2} = \frac{2^m}{2} = 2^{m-1}, \qquad n = 2^m$$

$$1 + \zeta_n^{\frac{n}{2}} = 1 + \zeta_n^{2^{m-1}} = 0, \qquad ---(4)$$

Therefore, in general:

$$f(\zeta_n) \quad = (1+\zeta_n)(1+\zeta_n{}^2)\cdots(\underline{\mathbf{1}+\boldsymbol{\zeta_n}^{\frac{n}{2}}})\cdots(1+\zeta_n{}^{n-1}) = \mathbf{0}$$

$$f(\zeta_n{}^2) \quad = (1+\zeta_n{}^2)(1+\zeta_n{}^4)\cdots(\underline{\mathbf{1}+\boldsymbol{\zeta_n}^{\frac{n}{2}}})\cdots(1+\zeta_n{}^{2n-2}) = \mathbf{0}$$

$$\vdots$$

$$f\left(\zeta_n{}^{\frac{n}{2}}\right) = (\underline{\mathbf{1}+\boldsymbol{\zeta_n}^{\frac{n}{2}}})(1+\zeta_n{}^n)\cdots(1+\zeta_n{}^{n-1(n)}) = \mathbf{0}$$

$$\vdots$$

$$f\left(\zeta_n{}^{n-1}\right) = (1+\zeta_n{}^{n-1})(1+\zeta_n{}^{2n-2})\cdots(\underline{\mathbf{1}+\boldsymbol{\zeta_n}^{\frac{n}{2}}})\cdots(1+\zeta_n{}^{(n-1)(n-1)}) = \mathbf{0}$$

Figure 17: Proposed generalization for all f(ζ_n), f($\zeta_n{}^2$),…, f($\zeta_n{}^{n-1}$) where n is a power of 2

The equivalent statement is that **for any $f(\zeta_n{}^{\mathrm{i}})$, at least one polynomial factor will evaluate to 0**.

Specifically, let P_m be the proposition that for every $m \in \mathbb{N}$ and $i \in \{1,2,\dots,n-1\}$, $f(\zeta_n{}^{\mathrm{i}}) = 0$.

We will proceed to prove this proposition via mathematical induction.

Verify P_m for the base cases P_1 and P_2:

For P_1:

$$n = 2^1 = 2$$

$$f(\zeta_2) = f(-1) = \left(\underline{1 + (-1)}\right) \cdots = 0$$

For P_2:

$$n = 2^2 = 4$$

$$f(\zeta_4) = f(i) = (1+i)\left(\underline{1 + (i^2)}\right) \cdots = (1+i)\left(\underline{1 + (-1)}\right) \cdots = 0$$

$$f\left({\zeta_4}^2\right) = f(\zeta_2) = f(-1) = \left(\underline{1 + (-1)}\right) \cdots = 0$$

$$f\left({\zeta_4}^3\right) = f(-i) = (1 + (-i))\left(\underline{1 + (-i^2)}\right) \cdots = (1+i)\left(\underline{1 + (-1)}\right) \cdots = 0$$

Assume P_k is true for some $k \in \mathbb{N}$:

$$n = 2^k$$

$$\zeta_{2^k} = e^{\left(\frac{i2\pi}{2^k}\right)} = e^{\left(\frac{i\pi}{2^{k-1}}\right)}$$

$$\frac{2^k}{2} = 2^{k-1}$$

$$\left(\zeta_{2^k}\right)^{2^{k-1}} = e^{\left(\frac{\left(2^{k-1}\right)i\pi}{2^{k-1}}\right)} = e^{i\pi} = -1$$

$$1 + \left(\zeta_{2^k}\right)^{2^{k-1}} = 1 - 1 = 0$$

$$f\left(\left(\zeta_{2^k}\right)^i\right) = \left(1 + \left(\zeta_{2^k}\right)^i\right)\left(1 + \left(\zeta_{2^k}\right)^{2i}\right) \cdots \left(\underline{1 + \left(\zeta_{2^k}\right)^{2^{k-1}}}\right) \cdots \left(1 + \left(\zeta_{2^k}\right)^{2ki}\right) = 0,$$

$$For\ every\ i_k \in \{1,2,\dots,2^k - 1\}.$$

Based on assumption P_k is true, prove P_{k+1} is true for every $i_{k+1} \in \{1,2,\dots,2^{k+1} - 1\}$:

$$\zeta_{2^{k+1}} = e^{\left(\frac{i2\pi}{2^{k+1}}\right)} = e^{\left(\frac{i\pi}{2^k}\right)}$$

$$\frac{2^{k+1}}{2} = 2^k$$

$$\left(\zeta_{2^{k+1}}\right)^{2^k} = e^{\left(\frac{(2^k)i\pi}{2^k}\right)} = e^{i\pi} = -1$$

$$1 + \left(\zeta_{2^{k+1}}\right)^{2^k} = 1 - 1 = 0$$

First consider all even values of $i_{k+1} = 2s$ where $s \in \{1,2, \ldots 2^k - 1\}$:

$$f\left(\left(\zeta_{2^{k+1}}\right)^{2s}\right) = f\left(\left(e^{\left(\frac{i\pi}{2^k}\right)}\right)^{2s}\right) = f\left(\left(e^{\left(\frac{2i}{2^k}\right)}\right)^{s}\right) = f\left(\left(\zeta_{2^k}\right)^s\right)$$

Since $s = i_k$,

$$f\left(\left(\zeta_{2^k}\right)^s\right) = \left(1 + \left(\zeta_{2^k}\right)^s\right)\left(1 + \left(\zeta_{2^k}\right)^s\right) \cdots \left(\underline{1 + \left(\zeta_{2^k}\right)^{2^{k-1}}}\right) \cdots \left(1 + \left(\zeta_{2^k}\right)^{2ks}\right) = 0$$

Holds true for all $i_{k+1} = 2s$.

Next consider all odd values of $i_{k+1} = 2s + 1$ where $s \in \{1,2, \ldots 2^k - 1\}$:

$$f\left(\left(\zeta_{2^{k+1}}\right)^{2s+1}\right)$$

Consider the polynomial factor $\left(1 + \left(\zeta_{2^{k+1}}\right)^{(2s+1)(2^k)}\right)$ with 2^k as an exponent to the root $\left(\zeta_{2^{k+1}}\right)^{2s+1}$ which is a term in every $f\left(\left(\zeta_{2^{k+1}}\right)^{2s+1}\right)$ due to the fact that **all odd numbers** (with the exception of the trivial value of 1) are **coprime to powers of 2**, **ensuring that all roots of the form $\left(\zeta_{2^{k+1}}\right)^{2s+1}$ are primitive**.

$$gcd\left((2s+1), 2^{k+1}\right) = 1$$

The value 2^k is also a valid choice as $2^k \in \{1,2, \ldots, 2^{k+1} - 1\}$ for $k \geq 0$.

$$1 + \left(\zeta_{2^{k+1}}\right)^{(2s+1)(2^k)} = 1 + \left(\underline{\left(\zeta_{2^{k+1}}\right)^{(2^k)}}\right)^{(2s+1)} = 1 + (-1)^{2s+1}$$

Since $2s + 1$ is odd, the polynomial factor evaluates to 0.

$$1 + (-1)^{2s+1} = 1 - 1 = 0$$

Since for both even and odd values of i_{k+1} which cover all values of i_{k+1}, the equations

$$f\left(\left(\zeta_{2^{k+1}}\right)^{2s}\right) = \left(1 + \left(\zeta_{2^k}\right)^{s}\right)\left(1 + \left(\zeta_{2^k}\right)^{2s}\right)\cdots\left(\underline{1 + \left(\zeta_{2^k}\right)^{2^{k-1}}}\right)\cdots\left(1 + \left(\zeta_{2^k}\right)^{2ks}\right) = 0$$

$$f\left(\left(\zeta_{2^{k+1}}\right)^{2s+1}\right) = \left(1 + \left(\zeta_{2^{k+1}}\right)^{2s+1)}\right)\left(1 + \left(\zeta_{2^{k+1}}\right)^{4s+2}\right)\cdots\left(\underline{1 + \left(\zeta_{2^{k+1}}\right)^{(2s+1)(2^k)}}\right)\cdots = 0$$

hold true, P_{k+1} is true.

Since P_1 and P_2 are true and P_k is true $\Longrightarrow$ P_{k+1} is true, by mathematical induction, the proposition P_m is true for all $m \in \mathbb{N}$.

Therefore, the solution for the number of subset-sums of {1,2,…,2021} divisible by powers of 2 is:

$$\sum c_{2^m \cdot k} = \frac{2^{2021}}{2^m}$$

3.2.2.2 General case

As established previously, when evaluating for composite numbers c, we not only have to consider the primitive c-th roots of unity but also the primitive roots of all its divisors.

Deriving a general closed-form solution for all composite divisors is highly convoluted and involves number theory beyond this paper's scope. Instead, we will demonstrate the solution process for an even composite ($c = 10$) and an odd composite ($c = 15$).

1. Break down c into its divisors

The divisors d of 10 are $1,2,5,10$. These divisors represent the order of the primitive roots (1st, 2nd, 5th and 10th) of unity present in the 10th roots of unity.

The divisors d of 15 are $1,3,5,15$. These divisors represent the order of the primitive roots (1st, 3rd, 5th and 15th) of unity present in the 15th roots of unity.

2. List out all c-th roots of unity

$$\zeta_{10}, {\zeta_{10}}^2, {\zeta_{10}}^3, {\zeta_{10}}^4, {\zeta_{10}}^5, {\zeta_{10}}^6, {\zeta_{10}}^7, {\zeta_{10}}^8, {\zeta_{10}}^9, {\zeta_{10}}^{10}$$

$$= \zeta_{10}, \underline{\zeta_5}, {\zeta_{10}}^3, \underline{{\zeta_5}^2}, \underline{\underline{\zeta_2}}, \underline{{\zeta_5}^3}, {\zeta_{10}}^7, \underline{{\zeta_5}^4}, {\zeta_{10}}^9, 1$$

$$\zeta_{15}, {\zeta_{15}}^2, {\zeta_{15}}^3, {\zeta_{15}}^4, {\zeta_{15}}^5, {\zeta_{15}}^6, {\zeta_{15}}^7, {\zeta_{15}}^8, {\zeta_{15}}^9, {\zeta_{15}}^{10}, {\zeta_{15}}^{11}, {\zeta_{15}}^{12}, {\zeta_{15}}^{13}, {\zeta_{15}}^{14}, {\zeta_{15}}^{15}$$

$$= \zeta_{15}, {\zeta_{15}}^2, \underline{\zeta_5}, {\zeta_{15}}^4, \underline{\underline{\zeta_3}}, \underline{{\zeta_5}^2}, {\zeta_{15}}^7, {\zeta_{15}}^8, \underline{{\zeta_5}^3}, \underline{\underline{{\zeta_3}^2}}, {\zeta_{15}}^{11}, \underline{{\zeta_5}^4}, {\zeta_{15}}^{13}, {\zeta_{15}}^{14}, 1$$

3. Utilize previously calculated values for $f(x)$ evaluated at the d-th roots of unity

$$f(1) = 2^{2021}, \qquad \text{from section 2.3}$$

$$f(\zeta_2) = f(-1) = 0, \quad \text{from section 2.3}$$

$$f(\zeta_5) + f({\zeta_5}^2) + f({\zeta_5}^3) + f({\zeta_5}^4) = 3(2^{404}), \quad \text{from section 2.5}$$

$$f(1) = 2^{2021}, \qquad \text{from section 2.3}$$

$$f(\zeta_3) + f({\zeta_3}^2) = 2(2^{673}), \quad \text{from appendix}$$

$$f(\zeta_5) + f({\zeta_5}^2) + f({\zeta_5}^3) + f({\zeta_5}^4) = 3(2^{404}), \quad \text{from section 2.5}$$

4. Evaluate $f(x)$ at the primitive c-th roots of unity

Recalling the previous result where A, $2A$, …, $(c-1)A$ are distinct modulo c when A and c are coprime, it would imply that for the product form of $f(\zeta_{10})$, $f({\zeta_{10}}^3)$, $f({\zeta_{10}}^7)$ and $f({\zeta_{10}}^9)$ there definitely exists a $(1+{\zeta_{10}}^5)$ term. Since ${\zeta_{10}}^5 = \zeta_2 = -1$, the $(1+{\zeta_{10}}^5)$ term evaluates to 0, causing (ζ_{10}), $f({\zeta_{10}}^3)$, $f({\zeta_{10}}^7)$ and $f({\zeta_{10}}^9)$ to also evaluate to 0. This holds true for every primitive c-th roots of unity **when c is even**, as earlier proven in (4).

As previously established for any $f(x)$ evaluated at primitive roots, the polynomial factors $(1+\zeta_{15})(1+{\zeta_{15}}^2)\cdots(1+{\zeta_{15}}^{15})$ will repeat across all $f(\zeta_{15})$, $f({\zeta_{15}}^2)$, $f({\zeta_{15}}^4)$, $f({\zeta_{15}}^7)$, $f({\zeta_{15}}^8)$, $f({\zeta_{15}}^{11})$, $f({\zeta_{15}}^{13})$ and $f({\zeta_{15}}^{14})$.

As such, when c is odd, we can apply result (3) for the value of $f(x)$ at a primitive root, where repeating polynomial factors $(1+\zeta_{15})(1+{\zeta_{15}}^2)\cdots(1+{\zeta_{15}}^{15})$ evaluate to 2, giving the same formula $2^q\left(\prod_j^r\left(1+{\zeta_p}^{ij}\right)\right)$. Hence the sum of all $f(x)$ at primitive roots **when c is odd** is given by:

$$2^q\sum_{i=1}^{c-1}\left(\prod_{j=1}^{r}\left(1+{\zeta_p}^{ij}\right)\right), \qquad where \ \gcd(i,c)=1$$

For $c=15$, the number of residual terms was substantial at $r=11$ (2021 = 134(15) + 11), rendering **manual calculations impractical**. Hence, a symbolic solver in python was used instead, for which this sum was evaluated to be $-6(2^{134})$.

5. Sum all evaluated values of $f(x)$ at the c-th roots of unity to obtain the solution

$$\sum c_{10k}=\frac{2^{2021}+0+3(2^{404})+0}{10}=\frac{2^{2021}+\ \ 3(2^{404})}{10}$$

In general, **when c is even**, the solution is simply the sum of $f(x)$ evaluated at the d-th roots of unity, which can be generalized to:

$$\sum c_{ck}=\frac{2^{2021}+\sum_{d=d_1}^{d_a}\left(\sum_{i=1}^{d-1}f\left({\zeta_d}^i\right)\right)}{c},$$

$$where\ d_1\ is\ the\ smallest\ divisor\ > 1\ and\ d_a\ is\ the\ largest\ divisor\ < c$$

$$\sum c_{15k} = \frac{2^{2021} + 2(2^{673}) + 3(2^{404}) - 6(2^{134})}{10}$$

In general, **when c is odd**, the solution is generalized to:

$$\sum c_{ck} = \frac{2^{2021} + \sum_{d=d_1}^{d_a}\left(\sum_{i=1}^{d-1} f\left({\zeta_d}^{i}\right)\right) + 2^q \sum_{k=1}^{c-1}\left(\prod_{j=1}^{r}\left(1 + {\zeta_p}^{kj}\right)\right)}{c},$$

$$where\ d_1\ is\ the\ smallest\ divisor\ > 1\ and\ d_a\ is\ the\ largest\ divisor\ < c,$$

$$and\ gcd(i, c) = 1$$

The term k was used in the final summation to prevent confusion with the index i used for the primitive d-th roots of unity.

The above process for composite divisors is multi-step, and the resulting general formulas are not explicit. Therefore, for a complete analysis of positive integer divisors, we **leverage computational tools to accelerate calculations**.

3.2.3 Algorithm for all positive integer divisors:

Our algorithmic process of evaluating the solution can be summarized in the flowchart below:

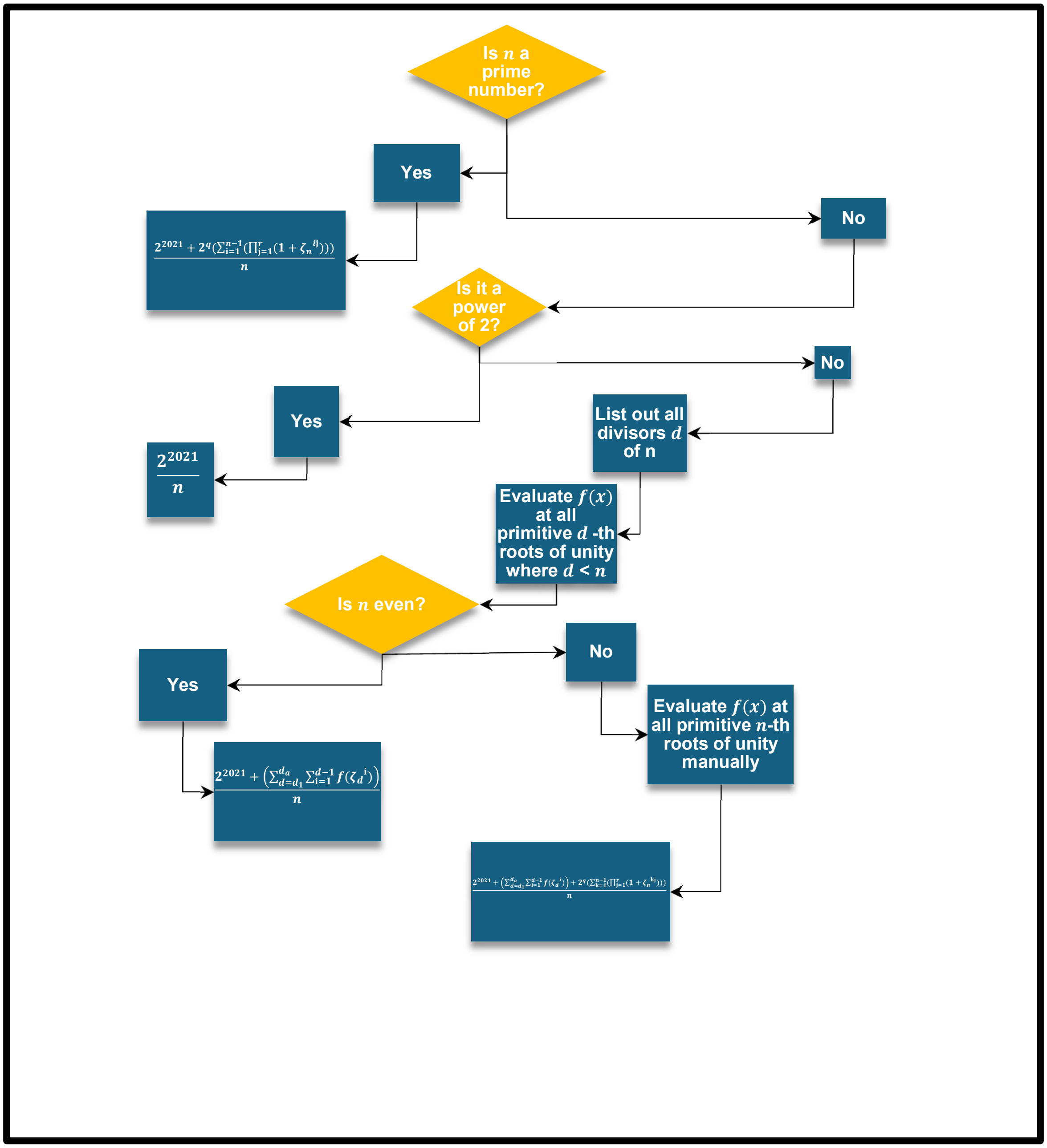

Figure 18: Flowchart algorithm for determining the number of subsets of {1,2,...,2021} divisible by n

Next, I **implemented the algorithm** outlined into the flowchart above **as executable python code** to generate the solutions [9] for the **first 100 positive integer divisors**.

[9] Output in Appendix

4. Evaluation

To analyse the distribution of the number of subset sums divisible by n for the set {1,2,…,2021}, the graph of the approximation $\frac{2^{2021}}{n}$ was plotted against the actual computed values for divisors 1,2,..,20 to allow for visual comparison using Python.

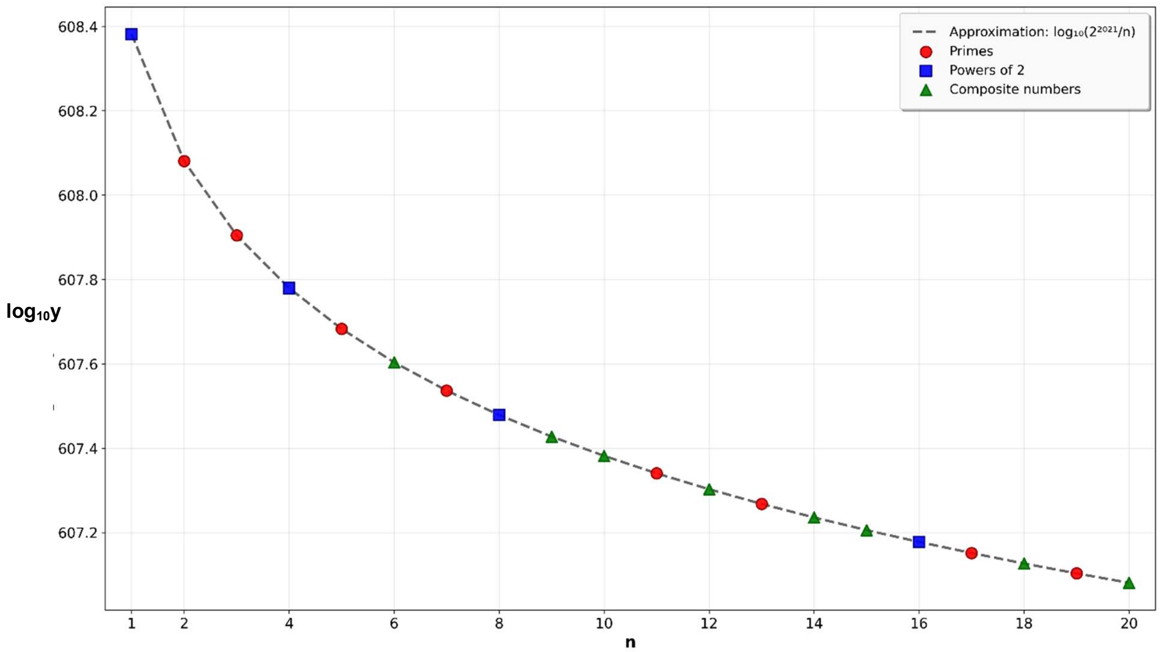

Figure 19: Graph of approximation $2^{2021}/n$ against the actual computed values for divisors 1,2,..,20

As observed from the graph above, $\frac{2^{2021}}{n}$ turns out to be a great approximation for small divisors, as all computed values fall on its curve. This aligns with a principle in additive combinatorics where **for large sets, subset sums tend towards a uniform distribution across all remainder classes modulo n** [10]. In the general formula $\frac{1}{n}\left(2^N + \left(\sum_{i=1}^{n-1} f(\zeta^i)\right)\right)$, 2^N represents the uniform distribution, while $\sum_{i=1}^{n-1} f(\zeta^i)$ represents the precise deviation from uniformity.

[10] Erdős, P., & Turán, P. (1948). On a problem in the theory of uniform distribution. I. Proceedings of the Koninklijke Nederlandse Akademie van Wetenschappen, 51, p. 1146-1154.

5. Conclusion:

In conclusion, the essay navigated a complex discrete combinatorics problem by geometrically framing it as the cycling of values on a unit circle, opening avenues for **leveraging the cyclic properties of roots of unity** within the complex plane. This was the pivotal breakthrough required to complement the powerful method of generating functions to elegantly cancel out irrelevant terms and isolate the coefficients of interest.

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All figures were digitally created by me in Microsoft word.

9. Appendix - Python code output for first 100 positive integer divisors

n = 1: (2^2021)/1

n = 2: (2^2021)/2

n = 3: (2^2021 + 2*2^673)/3

n = 4: (2^2021)/4

n = 5: (2^2021 + 3*2^404)/5

n = 6: (2^2021 + 2*2^673)/6

n = 7: (2^2021 + 3*2^288)/7

n = 8: (2^2021)/8

n = 9: (2^2021 + 2*2^673)/9

n = 10: (2^2021 + 3*2^404)/10

n = 11: (2^2021 + 8*2^183)/11

n = 12: (2^2021 + 2*2^673)/12

n = 13: (2^2021 + 2^155)/13

n = 14: (2^2021 + 3*2^288)/14

n = 15: (2^2021 + 2*2^673 + 3*2^404 - 6*2^134)/15

n = 16: (2^2021)/16

n = 17: (2^2021 + 8*2^118)/17

n = 18: (2^2021 + 2*2^673)/18

n = 19: (2^2021 + 5*2^106)/19

n = 20: (2^2021 + 3*2^404)/20

n = 21: (2^2021 + 2*2^673 + 3*2^288 - 18*2^96)/21

n = 22: (2^2021 + 8*2^183)/22

n = 23: (2^2021 + 17*2^87)/23

n = 24: (2^2021 + 2*2^673)/24

n = 25: (2^2021 + 3*2^404 - 25*2^80)/25

n = 26: (2^2021 + 2^155)/26

n = 27: (2^2021 + 2*2^673 - 18*2^74)/27

n = 28: (2^2021 + 3*2^288)/28

n = 29: (2^2021 + 6*2^69)/29

n = 30: (2^2021 + 2*2^673 + 3*2^404 - 6*2^134)/30

n = 31: (2^2021 - 33*2^65)/31

n = 32: (2^2021)/32

n = 33: (2^2021 + 2*2^673 + 8*2^183 - 173*2^61)/33

n = 34: (2^2021 + 8*2^118)/34

n = 35: (2^2021 + 3*2^404 + 3*2^288 + 16*2^57)/35

n = 36: (2^2021 + 2*2^673)/36

n = 37: (2^2021 - 5*2^54)/37

n = 38: (2^2021 + 5*2^106)/38

n = 39: (2^2021 + 2*2^673 + 2^155 - 7*2^51)/39

n = 40: (2^2021 + 3*2^404)/40

n = 41: (2^2021 + 947*2^49)/41

n = 42: (2^2021 + 2*2^673 + 3*2^288 - 18*2^96)/42

n = 43: (2^2021 + 42*2^47)/43

n = 44: (2^2021 + 8*2^183)/44

n = 45: (2^2021 + 2*2^673 + 3*2^404 - 6*2^134 - 60*2^44)/45

n = 46: (2^2021 + 17*2^87)/46

n = 47: (2^2021 + 46*2^43)/47

n = 48: (2^2021 + 2*2^673)/48

n = 49: (2^2021 + 3*2^288 + 553*2^41)/49

n = 50: (2^2021 + 3*2^404 - 25*2^80)/50

n = 51: (2^2021 + 2*2^673 + 8*2^118 - 76*2^39)/51

n = 52: (2^2021 + 2^155)/52

n = 53: (2^2021 - 75*2^38)/53

n = 54: (2^2021 + 2*2^673 - 18*2^74)/54

n = 55: (2^2021 + 3*2^404 + 8*2^183 - 119*2^36)/55

n = 56: (2^2021 + 3*2^288)/56

n = 57: (2^2021 + 2*2^673 + 5*2^106 + 191*2^35)/57

n = 58: (2^2021 + 6*2^69)/58

n = 59: (2^2021 + 9771*2^34)/59

n = 60: (2^2021 + 2*2^673 + 3*2^404 - 6*2^134)/60

n = 61: (2^2021 - 195*2^33)/61

n = 62: (2^2021 - 33*2^65)/62

n = 63: (2^2021 + 2*2^673 + 3*2^288 - 18*2^96 + 42*2^32)/63

n = 64: (2^2021)/64

n = 65: (2^2021 + 3*2^404 + 2^155 - 6*2^31)/65

n = 66: (2^2021 + 2*2^673 + 8*2^183 - 173*2^61)/66

n = 67: (2^2021 - 1981*2^30)/67

n = 68: (2^2021 + 8*2^118)/68

n = 69: (2^2021 + 2*2^673 + 17*2^87 - 77546*2^29)/69

n = 70: (2^2021 + 3*2^404 + 3*2^288 + 16*2^57)/70

n = 71: (2^2021 + 337*2^28)/71

n = 72: (2^2021 + 2*2^673)/72

n = 73: (2^2021 - 470*2^27)/73

n = 74: (2^2021 - 5*2^54)/74

n = 75: (2^2021 + 2*2^673 + 3*2^404 - 6*2^134 - 25*2^80 - 170*2^26)/75

n = 76: (2^2021 + 5*2^106)/76

n = 77: (2^2021 + 3*2^288 + 8*2^183 + 11758*2^26)/77

n = 78: (2^2021 + 2*2^673 + 2^155 - 7*2^51)/78

n = 79: (2^2021 + 30*2^25)/79

n = 80: (2^2021 + 3*2^404)/80

n = 81: (2^2021 + 2*2^673 - 18*2^74 - 216*2^24)/81

n = 82: (2^2021 + 947*2^49)/82

n = 83: (2^2021 - 610153*2^24)/83

n = 84: (2^2021 + 2*2^673 + 3*2^288 - 18*2^96)/84

n = 85: (2^2021 + 3*2^404 + 8*2^118 + 346*2^23)/85

n = 86: (2^2021 + 42*2^47)/86

n = 87: (2^2021 + 2*2^673 + 6*2^69 + 76983*2^23)/87

n = 88: (2^2021 + 8*2^183)/88

n = 89: (2^2021 + 11*2^22)/89

n = 90: (2^2021 + 2*2^673 + 3*2^404 - 6*2^134 - 60*2^44)/90

n = 91: (2^2021 + 3*2^288 + 2^155 + 215710*2^22)/91

n = 92: (2^2021 + 17*2^87)/92

n = 93: (2^2021 + 2*2^673 - 33*2^65 + 426*2^21)/93

n = 94: (2^2021 + 46*2^43)/94

n = 95: (2^2021 + 3*2^404 + 5*2^106 + 1910430*2^21)/95

n = 96: (2^2021 + 2*2^673)/96

n = 97: (2^2021 - 2301*2^20)/97

n = 98: (2^2021 + 3*2^288 + 553*2^41)/98

n = 99: (2^2021 + 2*2^673 + 8*2^183 - 173*2^61 - 572649*2^20)/99

n = 100: (2^2021 + 3*2^404 - 25*2^80)/100