The Monster That Broke Smoothness

An Investigation into the Weierstrass Function and the Limits of Calculus

By Mokshitha Chadalavada

I would like to thank you for taking the time to read this essay and consider the mathematical ideas it presents.

1 Introduction

For much of the nineteenth century, mathematicians believed that "nice" functions behaved like the curves they could sketch: continuous, mostly smooth, and differentiable except at a few isolated corners. This belief was not a formal theorem, but an assumption so deeply embedded in analysis that it shaped the way people proved results. Then, in 1872, Karl Weierstrass presented a function that was continuous everywhere and differentiable nowhere.

This function was quickly labelled "pathological" and "monstrous". Charles Hermite referred to such examples as a "lamentable scourge". What made the Weierstrass function so disturbing was not just that it broke intuition, but that it did so with surgical precision: it was built from familiar ingredients (cosine waves) arranged in a way that exposed a hidden weakness in the foundations of calculus. This essay investigates that function, proves its key properties in detail, and situates it within the broader story of smoothness, Taylor series, and fractal geometry.

2 From Smooth Curves to Pathologies

Classical analysis was dominated by functions like polynomials, exponentials and trigonometric functions. These are infinitely differentiable and can be represented locally by Taylor series. For example, the sine function admits the expansion

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots,$$

Figure above (A). Created by the author (2026).

which converges for all real x. This reinforced the belief that continuity and differentiability were closely linked: if a function was continuous, then at "most" points it should admit a derivative and a Taylor expansion.

Weierstrass' construction directly attacked this belief. He did not merely find a function with many non-differentiable points; he produced one with **no** differentiable points at all. Moreover, it was defined as a Fourier-type series, similar in spirit to the expansions used to model waves and heat flow. The scandal was not that the function was exotic, but that it was built from the very tools analysts trusted.

3 Defining the Weierstrass Function

Weierstrass' original example can be written as

$$W(x) = \sum_{n=0}^{\infty} a^n \cos\left(b^n \pi x\right), \qquad (1)$$

Figure (1). Created by the author (2026).

where $0 < a < 1$, b is a positive odd integer, and the parameters satisfy a growth condition such as

$$ab > 1 + \frac{\pi}{2}. \qquad (2)$$

Figure (2). Created by the author (2026).

Under these assumptions, $W(x)$ is continuous everywhere but differentiable nowhere.

Each term $a^n \cos(b^n \pi x)$ is a cosine wave of frequency b^n and amplitude a^n. As n increases, the waves become smaller in height but much more rapid in oscillation. The graph of $W(x)$ is the superposition of infinitely many such waves.

Figure 1: Visual appearance of a partial sum of the Weierstrass function

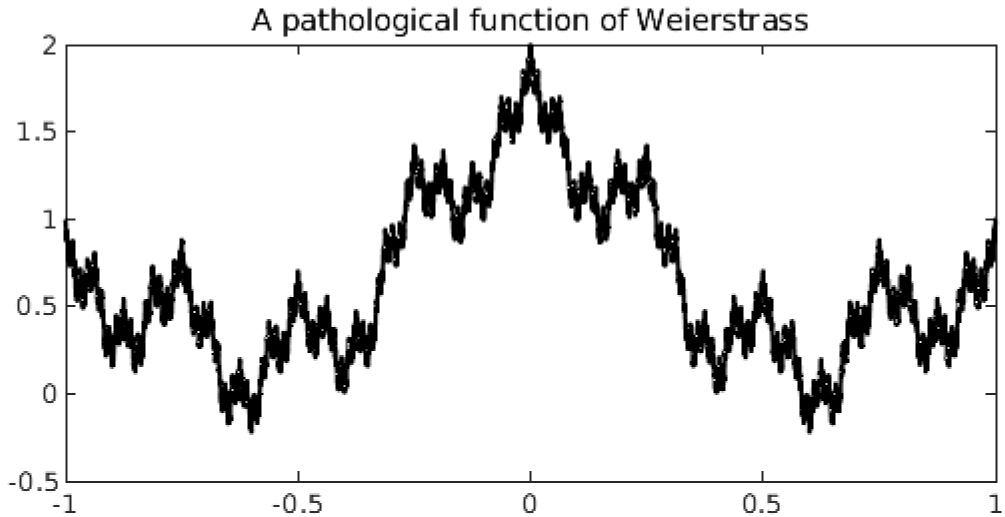

Figure 1: Graph of a partial sum of the Weierstrass function. **Source: Chebfun Project (n.d.). “A Pathological Function of Weierstrass.”**

$$W_N(x) = \sum_{n=0}^{N} a^n \cos\left(b^n \pi x\right)$$

Figure above (B). Created by the author (2026).

for $N = 1, 5, 50$ on the interval $[0,1]$. For small N, the curve resembles a slightly rough cosine. As N increases, finer oscillations appear at every scale. A zoomed-in view of a small subinterval (for example $[0.2,0.25]$) shows that the roughness persists under magnification.

These graphs illustrate visually what the analysis will prove: no matter how far one zooms in, the curve never becomes locally straight.

4 Continuity via Uniform Convergence

To show that $W(x)$ is continuous, we use the Weierstrass M-test. For each n,

$$|\, a^n \cos(b^n \pi x) \,| \leq a^n,$$

Figure above (C). Created by the author (2026).

and the geometric series converges because $0 < a < 1$. Hence the series in (1) converges uniformly on $\mathbb{R}$. Each partial sum $W_N(x)$ is continuous, and the uniform limit of continuous functions is continuous. Therefore, $W(x)$ is continuous everywhere.

This part is entirely standard: nothing "monstrous" appears yet. The pathology lies in the failure of differentiability.

5 Nowhere Differentiability: A Detailed Autopsy

To investigate differentiability, we examine the difference quotient at a point x:

$$Q(x,h) = \frac{W(x+h) - W(x)}{h}. \quad (3)$$

Figure (3). Created by the author (2026).

If W were differentiable at x, the limit of $Q(x,h)$ as $h \to 0$ would exist and be finite.

Substituting (1) into (3) gives

$$W(x+h)-W(x)=\sum_{n=0}^{\infty}a^n\left[\cos\left(b^n\pi(x+h)\right)-\cos\left(b^n\pi x\right)\right]. \qquad (4)$$

Figure (4). Created by the author (2026).

Using the trigonometric identity

$$\cos(u+v)-\cos(u)=-2\sin\left(\frac{v}{2}\right)\sin\left(u+\frac{v}{2}\right), \qquad (5)$$

Figure (5). Created by the author (2026).

with $u=b^n\pi x$ and $v=b^n\pi h$, we obtain

$$\cos\left(b^n\pi(x+h)\right)-\cos\left(b^n\pi x\right)=-2\sin\left(\frac{b^n\pi h}{2}\right)\sin\left(b^n\pi x+\frac{b^n\pi h}{2}\right). \qquad (6)$$

Figure (6). Created by the author (2026).

Thus

$$W(x+h)-W(x)=-2\sum_{n=0}^{\infty}a^n\sin\left(\frac{b^n\pi h}{2}\right)\sin\left(b^n\pi x+\frac{b^n\pi h}{2}\right). \qquad (7)$$

Figure (7). Created by the author (2026).

To expose the oscillatory behaviour, we choose a sequence of step sizes adapted to the frequencies. Following a standard approach, take

$$h=\frac{1}{4}b^{-N} \qquad (8)$$

Figure (8). Created by the author (2026).

for a large integer N. Then

$$\frac{b^n \pi h}{2} = \frac{\pi}{8} b^{n-N}. \quad (9)$$

Figure (9). Created by the author (2026).

For $n = N$, this becomes $\frac{\pi}{8}$, so

$$\sin\left(\frac{b^N \pi h}{2}\right) = \sin\left(\frac{\pi}{8}\right),$$

Figure above (D). Created by the author (2026).

a fixed positive constant. Meanwhile, for $n > N$, the argument $\frac{\pi}{8} b^{n-N}$ is a multiple of $\frac{\pi}{2}$ and the periodicity of the sine and cosine functions implies that those higher-frequency terms cancel exactly in the difference $W(x+h) - W(x)$.

Thus, for this special choice of h, the infinite sum in (7) reduces to a finite sum from $n = 0$ to $n = N$. The $n = N$ term will be the “main term”, and the others will be treated as an error.

We write

$$W(x+h) - W(x) = T_N(x,h) + R_N(x,h), \quad (10)$$

Figure (10). Created by the author (2026).

where

$$T_N(x,h) = -2a^N \sin\left(\frac{b^N \pi h}{2}\right) \sin\left(b^N \pi x + \frac{b^N \pi h}{2}\right) \quad (11)$$

Figure (11). Created by the author (2026).

is the $n = N$ term, and R_N is the sum of the remaining terms with $0 \leq n < N$.

Dividing by h and using (8), we obtain

$$Q(x,h) = \frac{W(x+h) - W(x)}{h} = \frac{T_N(x,h)}{h} + \frac{R_N(x,h)}{h}. \quad (12)$$

Figure (12). Created by the author (2026).

The magnitude of the main term satisfies

$$\left| \frac{T_N(x,h)}{h} \right| = \frac{2a^N}{|h|} \left| \sin\left(\frac{b^N \pi h}{2}\right) \right| \left| \sin\left(b^N \pi x + \frac{b^N \pi h}{2}\right) \right|. \quad (13)$$

Figure (13). Created by the author (2026).

Using $|\sin(\theta)| \leq 1$ and $|\sin(\pi/8)| > 0$, and substituting $|h| = \frac{1}{4}b^{-N}$, we obtain

$$\left| \frac{T_N(x,h)}{h} \right| \geq C_1 a^N b^N = C_1 (ab)^N, \quad (14)$$

Figure (14). Created by the author (2026).

for some positive constant C independent of N.

Next, we bound the error term. For $n < N$, we have

$$| \sin\left(\frac{b^n \pi h}{2}\right) | \leq | \frac{b^n \pi h}{2} | = \frac{\pi}{8} b^{n-N}, \quad (15)$$

Figure (15). Created by the author (2026).

using $|\sin t| \leq |t|$. Also, $|\sin(\cdot)| \leq 1$ for the second sine factor. Hence

$$|R_N(x,h)| \leq 2\sum_{n=0}^{N-1} a^n | \sin\left(\frac{b^n \pi h}{2}\right) | \leq 2\sum_{n=0}^{N-1} a^n \frac{\pi}{8} b^{n-N} = C_2 b^{-N} \sum_{n=0}^{N-1} (ab)^n, \quad (16)$$

Figure (16). Created by the author (2026).

for some constant C_2. Dividing by $|h| = \frac{1}{4} b^{-N}$ gives

$$| \frac{R_N(x,h)}{h} | \leq C_3 \sum_{n=0}^{N-1} (ab)^n, \quad (17)$$

Figure (17). Created by the author (2026).

with another constant C_3.

The sum on the right is a geometric series:

$$\sum_{n=0}^{N-1} (ab)^n = \frac{(ab)^N - 1}{ab - 1}. \quad (18)$$

Figure (18). Created by the author (2026).

We now compare the main term and the error using the reverse triangle inequality:

$$\left| \frac{W(x+h)-W(x)}{h} \right| \geq \left| \frac{T_N(x,h)}{h} \right| - \left| \frac{R_N(x,h)}{h} \right|. \quad (19)$$

Figure (19). Created by the author (2026).

Substituting (14) and (17)–(18), we obtain

$$\left| \frac{W(x+h)-W(x)}{h} \right| \geq C_1(ab)^N - C_3\frac{(ab)^N-1}{ab-1}. \quad (20)$$

Figure (20). Created by the author (2026).

Because $ab > 1 + \frac{\pi}{2} > 1$, the term $(ab)^N$ grows exponentially. For sufficiently large N, the right-hand side of (20) becomes arbitrarily large in magnitude. In particular, the sequence of difference quotients $Q(x, h)$ along the steps $h = \frac{1}{4}b^{-N}$ is unbounded.

Therefore, the limit of $Q(x, h)$ as $h \to 0$ cannot exist at any point x. We conclude:

The Weierstrass function $W(x)$ is continuous everywhere and differentiable nowhere.

This is not merely a sketch: the main term has been isolated, the remaining terms have been bounded, and the reverse triangle inequality has been used to show that no cancellation can rescue differentiability.

6 Smoothness, Taylor Series, and Failure

For classical functions like $sin\ x$, differentiability implies the existence of a Taylor series expansion around any point:

$$\sin x = \sum_{k=0}^{\infty} \frac{\sin^{(k)}(0)}{k!} x^k.$$

Figure above (E). Created by the author (2026).

The derivatives $sin^{(k)}(0)$ are well-defined and finite, and the series converges to $sin\ x$ for all x.

For the Weierstrass function, the situation is radically different. At any point x_0 , the derivative $W'(x_0)$ does not exist. Hence there is no Taylor series expansion of the form

$$W(x) = \sum_{k=0}^{\infty} \frac{W^{(k)}(x_0)}{k!} (x - x_0)^k,$$

Figure above (F). Created by the author (2026).

because even the first derivative fails to exist. The function is continuous but has no local polynomial approximation in the usual sense. This is precisely what made it so shocking: it looked like a “curve”, but calculus could not linearise it at any point.

7 Fractal Dimension and Roughness

The graph of $W(x)$ is not just non-differentiable; it is fractal. Its Hausdorff dimension lies strictly between 1 and 2. For functions of the form (1), one can show that the dimension D of the graph satisfies

$$D = 2 + \frac{\ln a}{\ln b}, \quad (21)$$

Figure (21). Created by the author (2026).

which is greater than 1 because $1na < 0$ and $1nb > 0$.

Intuitively, the graph is “thicker” than a smooth curve but does not fill an area like a surface. This connects the Weierstrass function to later developments in fractal geometry, where such non-integer dimensions became a central concept.

Figure 2 (conceptual description). A sequence of zoomed-in plots of the graph of $W(x)$ on smaller and smaller intervals, each rescaled to the same width. The visual appearance remains jagged at every scale, illustrating self-similarity and supporting the idea of a fractal dimension greater than 1.

8 Mathematics versus Physical Reality

From a purely mathematical standpoint, the Weierstrass function is perfectly legitimate. From a physical standpoint, it is impossible. Real-world signals cannot oscillate arbitrarily fast with non-vanishing amplitude: materials have inertia, energy is finite, and there is a smallest meaningful length scale (often associated with the Planck length in physics).

Natural fractals—coastlines, clouds, tree branches—exhibit self-similar behaviour only across a finite range of scales. At very small scales, the structure smooths out or becomes discrete. The Weierstrass function, by contrast, is infinitely jagged at all scales. It is a model of “ideal roughness”, useful for theory but unattainable in nature.

This tension is mathematically ironic: the shapes that look most like natural phenomena (fractals) were initially dismissed as “unnatural” and “pathological”, while the perfectly smooth functions of classical analysis are the ones nature rarely realises exactly.

9 Conclusion

The Weierstrass function marks a turning point in the history of analysis. By constructing a function that is continuous everywhere and differentiable nowhere, Weierstrass forced mathematicians to abandon intuition based on smooth curves and to rebuild calculus on rigorous definitions and proofs. The detailed analysis of its difference quotients shows precisely how and why differentiability fails: the oscillations at different scales cannot be tamed by limits, and no Taylor series can capture its behaviour.

What began as a "monster" later became a prototype. Its graph anticipated the fractal curves of Mandelbrot, the irregular paths of Brownian motion, and the complex signals studied in modern physics and engineering. The Weierstrass function is not just a counterexample; it is a bridge between classical calculus and the geometry of roughness.

In that sense, it is more than a curiosity. It is a reminder that even the most trusted ideas in mathematics can be challenged by a single, carefully constructed example—and that true understanding often begins where intuition fails.

References

- Weierstrass, K. (1872). *Über continuirliche Functionen ohne definirte Ableitung*.
- Hardy, G. H. (1916). *Weierstrass's Non-Differentiable Function*.
- Mandelbrot, B. (1982). *The Fractal Geometry of Nature*.
- Falconer, K. (2014). *Fractal Geometry: Mathematical Foundations and Applications*. Johnsen, J. (2010). "Simple proofs of nowhere-differentiability for Weierstrass's function and cases of slow growth", *Journal of Fourier Analysis and Applications*.
- University of Pennsylvania (n.d.). *Advanced Analysis: Continuous, Nowhere Differentiable Functions* (lecture notes).
- Chebfun Project (n.d.) *A Pathological Function of Weierstrass*. Available at: https://www.chebfun.org (Accessed: 12 April 2026).