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How would GPS be affected by different shaped earths

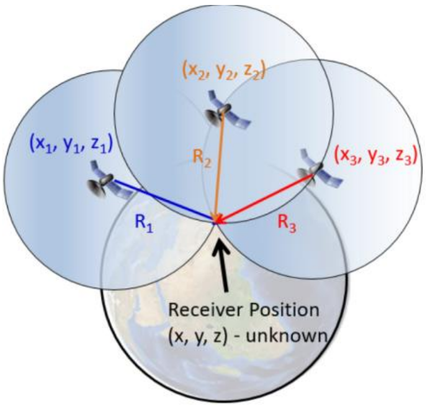
INTRODUCTION

The global positioning system, or GPS for short, is something that has become increasingly important in our lives today. Whether it's tracking how long your runs are, or navigating your way through a foreign city, there is always a use for our satellite friends, or foes- but that's another matter entirely.

GPS comes down to one core aspect, trilateration, which is the concept of determining position by measuring distances from several reference points. In space, GPS satellites broadcast signals that allow a receiver on Earth to compute how far away each satellite is. These distances define spheres in three dimensions; the exact location of the receiver is found where at least four such spheres intersect. The basic trilateration equation for a single satellite can be written as:

$$(x - x_i)^2 + (y - y_i)^2 + (z - z_i)^2 = r_i^2$$

It is visualised here:



where (xi, yi, and zi) are the known coordinates of satellite (i) and (ri) is the measured distance from the satellite to the receiver. On paper, this looks straightforward, but what we often overlook is the subtle dependence of this calculation on the assumed shape of the Earth. A slight change in the Earth's model changes the geometry of these spheres and the final calculated position. In the following paragraphs, we explore how GPS accuracy changes when we imagine different shapes of Earth: our real oblate spheroid (and geoid), a toroidal Earth, and a flat Earth.

OUR CURRENT EARTH MODEL

The assumption that the Earth is a perfect sphere is a useful starting point, but it is not accurate enough for a system as precise as GPS. In reality, the Earth is better described as an oblate spheroid, meaning it is slightly flattened at the poles and bulges at the equator. This can be modelled mathematically by the equation:

$$\frac{x^2 + y^2}{a^2} + \frac{z^2}{b^2} = 1$$

where a is the equatorial radius and b is the polar radius. For Earth, these values are approximately a=6378.1 km and b=6356.8 km. Although the difference between these values is relatively small, it becomes significant when calculating positions across large distances.

GPS satellites orbit at an altitude of about 20,200 km, giving them a distance of roughly 26,571 km from the centre of the Earth. When a receiver calculates its position using the trilateration equation it must take into account the fact that the Earth's surface does not lie on a perfect sphere. The curvature changes slightly depending on latitude, which affects how distances along the surface are interpreted. This is corrected using quantities such as the radius of curvature, given by:

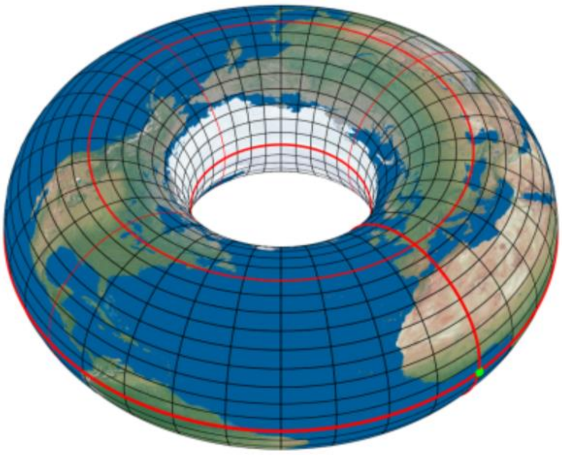
$$M = \frac{a(1-e^2)}{(1-e^2\sin^2\phi)^{\frac{3}{2}}} \text{ where } e^2 = \frac{a^2-b^2}{a^2}$$

Here, ϕ represents latitude, and e^2 measures how much the Earth deviates from a perfect sphere. Without applying these corrections, even slight differences in curvature would lead to noticeable errors in position.

Beyond the ellipsoid model, GPS must also account for the geoid, which represents variations in Earth's gravitational field. This affects how height is measured, since "sea level" is not uniform across the planet. By combining the ellipsoid for geometry and the geoid for gravity, GPS systems are able to achieve the level of accuracy we rely on today.

TORUS EARTH

To understand how strongly GPS depends on the Earth's shape, it is useful to consider a more unusual example: a toroidal Earth, shaped like a doughnut.



Although this is purely hypothetical, it provides an interesting mathematical comparison. A torus can be described using the parametric equation (for simplicity) :

$$x = (R + rcos\theta)cos\phi, \; y = (R + rcos\theta)cos\phi, \; z = rsin\theta$$

where R is the distance from the centre of the torus to the centre of the tube, and r is the radius of the tube itself. Unlike a sphere or ellipsoid, the curvature of a torus is not consistent. The outer region curves gently, while the inner region curves much more sharply in the opposite direction.

This variation can be visualised in the diagram, where paths drawn across the surface wrap around the torus in different ways. Some paths loop around the central hole, while others follow the outer surface. From a GPS perspective, this creates a key problem. The trilateration equation

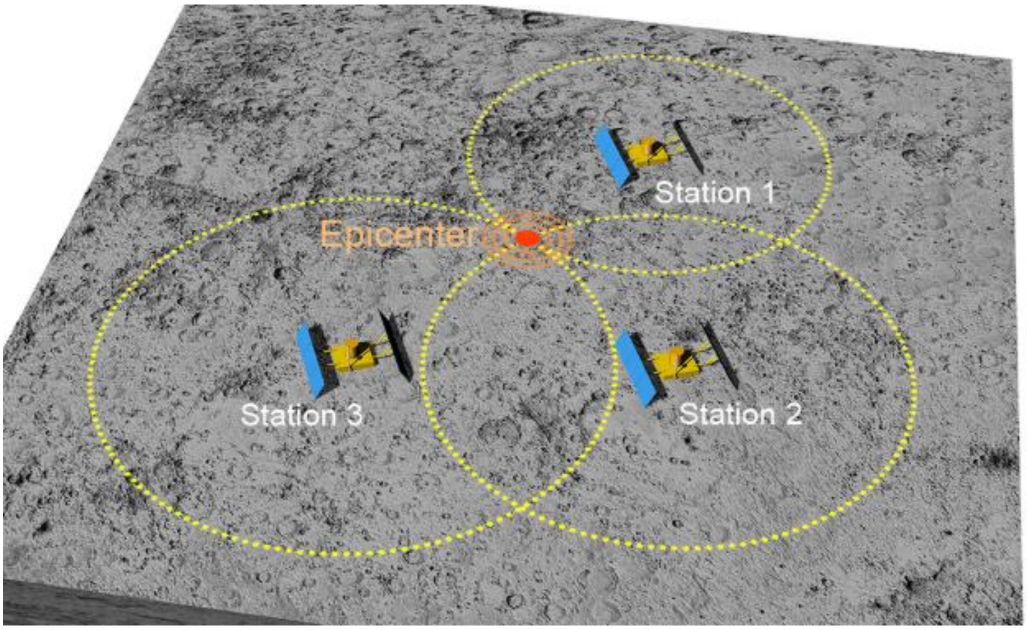
assumes that a given set of distances corresponds to a unique position. However, on a torus, distances measured from satellites could correspond to multiple points that lie along different routes around the surface. For example, a position on the outer edge and a position closer to the inner ring could produce similar distance measurements, especially when signals effectively "wrap around" the geometry of the torus.

In addition, the paths shown in the diagram highlight that movement across a toroidal surface is not straightforward. Travelling in what appears to be a straight direction can lead back to the starting point in a non-intuitive way, depending on how the path loops around the torus. This would further complicate GPS calculations, as the system would need to distinguish between multiple possible routes and positions. As a result, even though the underlying equations remain the same, the geometry of a torus makes reliable positioning far more difficult.

FLAT EARTH

A final model to consider is a flat Earth, where the surface is treated as a two-dimensional plane rather than a curved object. In this case, the standard GPS system would need to be redesigned to work as effectively as possible under these new geometric constraints. Trilateration would take the simpler two-dimensional form:

$$(x - x_i)^2 + (y - y_i)^2 = r_i^2$$



It would work similarly to earthquake trilateration, which uses intersections to find the epicentre of earthquakes.

suggesting that, in theory, only three satellites would be needed to determine a position on the plane. However, because real satellites exist in three-dimensional space, an optimal setup would require careful positioning to minimise distortion when projecting their signals onto the flat surface.

One possible configuration would be to arrange satellites symmetrically above the plane, maintaining equal spacing and similar altitudes. This would ensure that distance measurements remain as consistent as possible across the surface. In such a system, the most accurate results would occur near the centre of the satellite network, where the geometry is most balanced. This is similar to how trilateration works best when the reference points surround the receiver evenly.

However, even in this optimal arrangement, problems begin to appear as one moves further from the centre. The lack of curvature means that distances derived from satellites no longer align naturally with positions on the surface. To compensate, additional correction terms would need to be introduced, effectively modifying the distance equation to account for projection error:

$$(x - x_i)^2 + (y - y_i)^2 = r_i^2 + \epsilon(x, y)$$

where $\epsilon(x,y)$ represents a position-dependent error term. This term would grow larger towards the edges of the plane, reflecting the increasing mismatch between the true three-dimensional distances and their two-dimensional interpretation.

Altitude presents a further challenge. In a flat Earth model, height cannot be defined relative to a central point in the same way as on a sphere or ellipsoid. An optimal system would therefore need an additional layer of satellites or a separate vertical reference system to determine height accurately.

Overall, while an optimised satellite arrangement could reduce some errors locally, the flat Earth model requires increasingly complex corrections to maintain accuracy. This highlights that it is not just the equations themselves that matter, but the underlying geometry they are applied to.

CONCLUSION

Bringing everything together, it’s clear that GPS isn’t just about satellites, but about how accurately we model the Earth. The idea of trilateration stays the same in every case, but whether it actually works depends on the shape we apply it to. For the real Earth, using an oblate spheroid and making small corrections means the maths lines up well with reality, which is why GPS is so accurate in practice.

When you look at different shapes, the problems become much clearer. On a torus, the same set of distances could give more than one possible location, which makes it hard to know where you actually are. On a flat Earth, even if you try to arrange the satellites in the best possible way, the errors still grow as you move further out, and the system becomes less reliable. In both cases, the equations themselves haven’t changed, but the geometry they rely on has.

Overall, this shows that even small assumptions in maths can make a big difference. By choosing a model of the Earth that is as close to reality as possible, GPS is able to work as well as it does. Without that, even the right equations would give the wrong answers.

Thank you for reading.