67 - Prime Perfection

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1 Introduction

In digital culture, 67 is often perceived as a memetic placeholder. Yet, beneath this facade of internet irony lies rigorous and eloquent algebraic structure. With the distribution of prime numbers being famously chaotic, this specific integer serves as the foundation of arithmetic stability that Leonhard Euler first began to discover in the 18th century.

2 Euler's breakthrough

In the late 1700s, a group of quadratic functions caught Euler's eye due to their unique behaviour in prime generation. Defining the function:

$$f(x) = x^2 + x + 17$$

We get a quadratic function with complex roots:

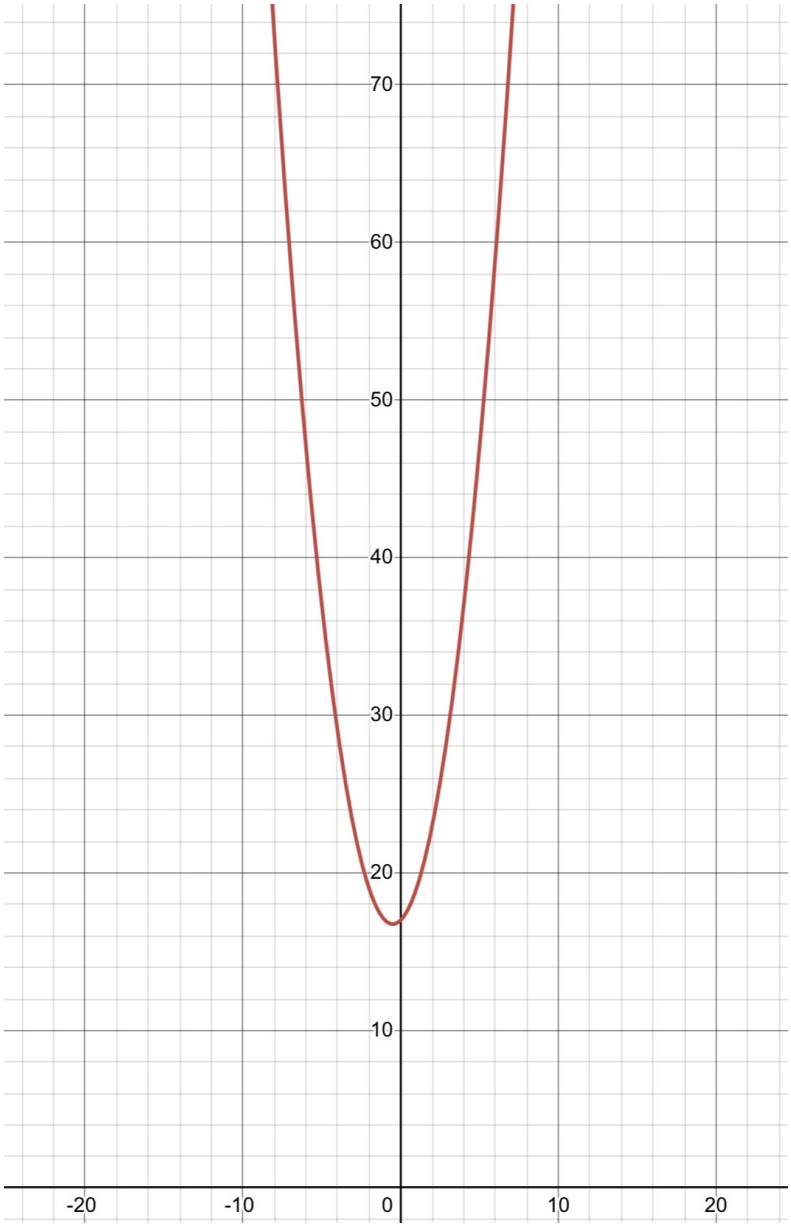

Figure 1: $f(x) = x^2 + x + 17$.

Table 1: Values of $f(x)$ for $0 \leq x \leq 17$.

x	$f(x)$
0	17
1	19
2	23
3	29
4	37
5	47
6	59
7	73
8	89
9	107
10	127
11	149
12	173
13	199
14	227
15	257
16	289
17	323

Remarkably, $f(x)$ generates consecutive primes $\forall x$ up to 15, failing at $f(16) = 17^2$. Although 67 isn't in the list, it remains tied to this prime-lineage as a Chen prime and the sum of five consecutive primes $(7 + 11 + 13 + 17 + 19)$. Seeming as a coincidence, 67 acts as the structural backbone for the very patterns Euler spent years uncovering.

Chen primes: $\{p \in \mathbb{P} \mid p + 2 \in \mathbb{P} \vee p + 2 \in P_2\}$, P_2 *are semi-primes*

Moreover, we see its discriminant:

$$D = (1)^2 - 4(1)(17)$$
$$D = -67$$

This initial encounter with -67 is no coincidence; its significance lies in D.

3 67 and its tight friend group

To understand 67, we must examine its mathematical 'neighborhood'. While $\mathbb{Z}$ rely on UF - the foundation of arithmetic - mathematicians also explore alternative 'worlds' called Number Fields. Our focus is $\mathbb{Q}(\sqrt{-67})$: $\mathbb{Q}$ injected with $\sqrt{-67}$. In this field, regular whole numbers are replaced by algebraic integers, denoted as $\mathcal{O}_K$.

> *$\mathcal{O}_K$: Complex roots of monic polynomials - those with integer coefficients and a leading coefficient of 1. In the field $\mathbb{Q}(\sqrt{-d})$ where $-d \equiv 1 \pmod 4$, these integers take the form $\frac{a+b\sqrt{-d}}{2}$, provided a and b share the same parity.*

Since $-67 \equiv 1 \pmod 4$, algebraic integers in this field take the form:

$$\mathcal{O}_K = \left\{ \frac{a + b\sqrt{-67}}{2} \mid a, b \in \mathbb{Z}, a \equiv b \pmod 2 \right\}$$

This world allows us to explore the class number (h), which essentially measures how much the laws of unique factorisation (UF) are broken. In most of these worlds, mathematical computation becomes messy. Take the number 6 in the world $\mathbb{Q}(\sqrt{-5})$ where $h = 2$.

$$6 = 2 \cdot 3$$
$$\text{and}$$
$$6 = (1 + \sqrt{-5}) \cdot (1 - \sqrt{-5})$$

Suddenly, 6 has two UFs. However there are only nine special negative discriminants where this law of UF remains intact: the Heegner numbers.

The Heegner numbers are: 1, 2, 3, 7, 11, 19, 43, 67, and 163.

Every Heegner number has $h = 1$, meaning in the complex world $\mathbb{Q}(\sqrt{-67})$, every number still has a single UF. Take 17, while prime on our standard world, in $\mathbb{Q}(\sqrt{-67})$ it can be factored into half 'whole numbers' of the field:

$$17 = \left(\frac{1 + \sqrt{-67}}{2} \right) \left(\frac{1 - \sqrt{-67}}{2} \right)$$

As defined earlier, for a number α in this world, it is factorised in the form:

$$\alpha = \left(\frac{a + b\sqrt{-67}}{2} \right) \left(\frac{a - b\sqrt{-67}}{2} \right)$$

Therefore:

$$\alpha = \frac{a^2 + 67b^2}{4}$$

For example, equating this to 19:

$$\frac{a^2 + 67b^2}{4} = 19 \implies a^2 + 67b^2 = 76$$

We can find values a and b such that they have the same parity. If we set $b = 1$:

$$a^2 + 67(1)^2 = 76$$
$$a^2 + 67 = 76$$
$$a^2 = 9 \implies a = \pm 3$$

Since $a = 3$ and $b = 1$ are odd, they satisfy the parity condition for an algebraic integer in this field. Thus, the UF of 19 in $\mathbb{Q}(\sqrt{-67})$ is:

$$19 = \left(\frac{3 + \sqrt{-67}}{2}\right)\left(\frac{3 - \sqrt{-67}}{2}\right)$$

In a world with $h > 1$, we can find many ways to factorise 19. But as 67 is a Heegner number, this specific pairing is the *only* way to factor it. This exciting structural ability makes 67 a number of absolute perfection in any world of mathematics!

Table 2: UF examples in $\mathbb{Q}(\sqrt{-67})$.

Integer	**UF in** $\mathbb{Q}(\sqrt{-67})$
17	$\left(\frac{1+\sqrt{-67}}{2}\right)\left(\frac{1-\sqrt{-67}}{2}\right)$
19	$\left(\frac{3+\sqrt{-67}}{2}\right)\left(\frac{3-\sqrt{-67}}{2}\right)$
23	$\left(\frac{5+\sqrt{-67}}{2}\right)\left(\frac{5-\sqrt{-67}}{2}\right)$
29	$\left(\frac{7+\sqrt{-67}}{2}\right)\left(\frac{7-\sqrt{-67}}{2}\right)$
37	$\left(\frac{9+\sqrt{-67}}{2}\right)\left(\frac{9-\sqrt{-67}}{2}\right)$
47	$\left(\frac{11+\sqrt{-67}}{2}\right)\left(\frac{11-\sqrt{-67}}{2}\right)$

Anlysing this table, we see something more exciting! For any prime generated by $f(x) = x^2 + x + 17$, its UF in $\mathbb{Q}(\sqrt{-67})$ always uses $a = 2x + 1$ and $b = 1$. This explicitly shows us that the 'lucky' nature of $f(x)$ is directly tied to the fact that $D = -67$ allows for such clean UFs.

4 The Bridge

How do prime generating functions and Heegner numbers - both linked by the number 67 - connect? Well, the mathematician Rabinowitsch provided the 'missing link', proving that these functions are fundamentally dependent on Heegner numbers. His theorem established that a quadratic $x^2 + x + p$ generates consecutive primes if and only if its discriminant $D = 1 - 4p$ has a class number of $h = 1$. This implies that D *must* be a negative Heegner number, revealing that prime distribution is controlled by the field's class number, $h(D)$.

To understand why one of Euler's certain functions - $f(x) = x^2 + x + 17$ - works, let's work through the formal proof:

- Let H be the set of Heegner numbers.
- Let p be a prime number.
- $f(x) := x^2 + x + p \therefore D = 1 - 4p$.

Therefore, we have the given statement (*):

> *The function $f(x)$ will generate prime numbers for all $x \in \mathbb{Z}$ such that $0 \leq x \leq p-2$ if and only if the field $\mathbb{Q}(\sqrt{D})$ (or $\mathbb{Q}(\sqrt{1-4p})$) yields a class number $h = 1$.*

Testing this theorem using one of Euler's prime generating functions:

$$\begin{aligned} p &= 17 \\ D &= 1 - 4(17) \\ D &= -67 \end{aligned}$$

This points us to the specific number field $\mathbb{Q}(\sqrt{-67})$, and as we have established before, $67 \in H$. Thus by definition:

$$h(-67) = 1$$

Implying the law of UF is maintained in its world of algebraic integers.

Furthermore, substituting $p = 17$ into the bounds of x:

$$\begin{aligned} \max(x) &= (17) - 2 \\ \max(x) &= 15 \end{aligned}$$

Consequently, the theorem predicts failure at $x = p - 1 = 16$:

$$\begin{aligned} f(16) &= (16)^2 + 16 + 17 \\ &= 289 \\ &= 17^2 \implies f(16) \notin \mathbb{P} \end{aligned}$$

This result is perfectly satisfactory.

A polynomial function that is well known for this theory is:

$$f(x) = x^2 + x + 41$$

It impressively outputs 40 consecutive primes!

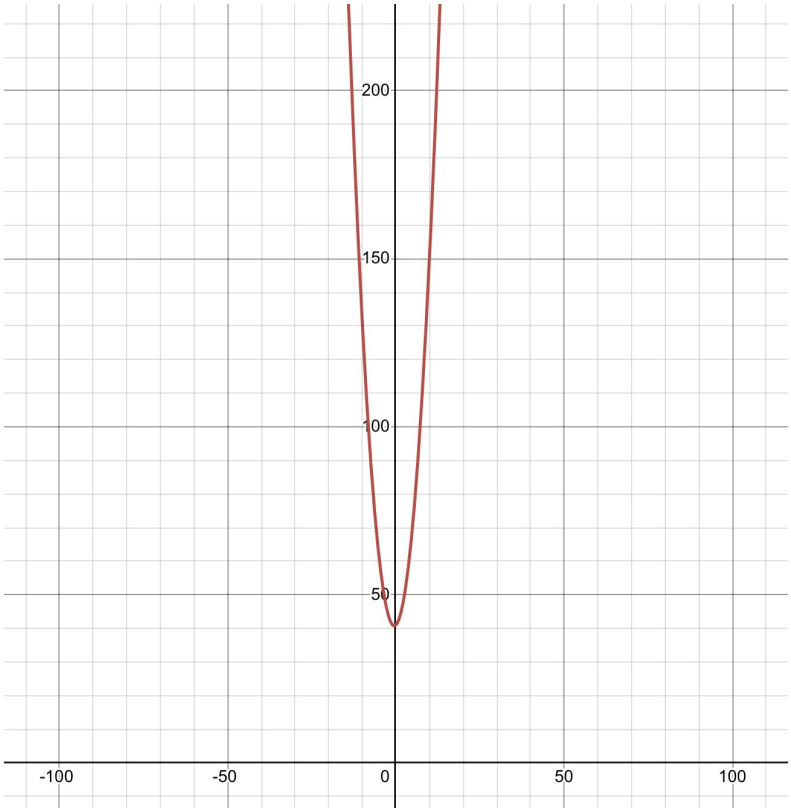

Figure 2: $f(x) = x^2 + x + 41$.

67 belongs to an important group of numbers. We clearly see that 67 isn't just a meaningless joke, but the key basis for such beautiful mathematical computation.

5 A lucky 67 link

With condition (*), computing gets us the functions:

$$f(x) = x^2 + x + p, \quad p \in P$$

$$P = \{1, 2, 3, 5, 11, 17, 41\}$$

Notably, the valid values for p correspond to Euler's Lucky numbers, where 67 again appears as a pivotal link. These integers are 'lucky' as they underpin the existence of prime generating quadratics. Euler demonstrated that these numbers allow for specialised functions! It is as follows:

- Let L be the set of Euler's lucky numbers: $L = \{2, 3, 5, 11, 17, 41\}$.
- $g(x) := x^2 - x + k$, $k \in L$ and $1 \leq x < k$, $g(x) \in \mathbb{P}$.

We now have two groups of prime generating functions!:

$$f(x) = x^2 + x + p$$

and

$$g(x) = x^2 - x + k$$

We can see they are reflections of each other in the y-axis when $p \in L$ as $L \subset P$:

$$f(x) = g(-x), \quad g(x) = f(-x) \implies \text{reflection in the } y\text{-axis}$$

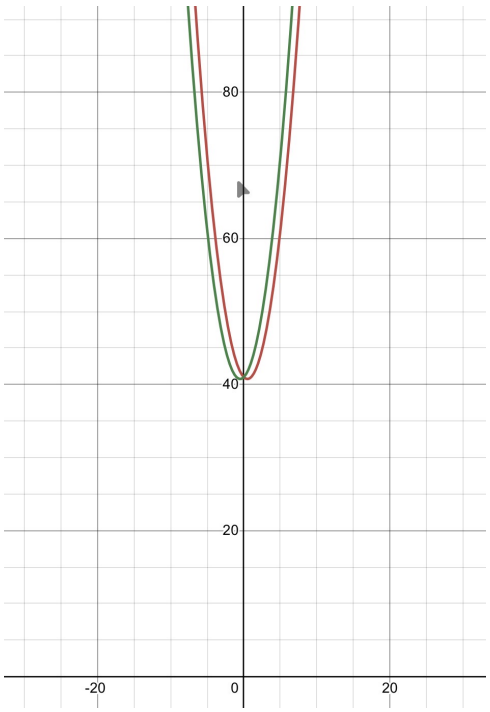

Figure 3: Symmetry and reflection of $f(x)$ and $g(x)$ across the y-axis.

Even though Rabinowitsch formulated his theory for $f(x)$, $g(x)$ still follows it as when the constants of both functions take values from $P \cap L$, their discriminants will both be the same, thus being a negative Heegner number.

6 π, e and 67

The pure elegance of 67 and Heegner numbers extend beyond prime generation into the realm of transcendental numbers. When combined with π and e, specific Heegner numbers produce 'near-integers'! We can explore this phenomenon by defining the function $r(x)$:

$$r(x) = e^{\pi\sqrt{x}}, \quad x \in H$$

One may assume that $r(x)$ can never output a 'clean' result as π, e, and $\sqrt{d}$ are all irrational. However, computation uncovers an intriguing pattern. $\forall d \in H$, evaluating $r(d)$:

```
from mpmath import mp

mp.dps = 50

H = [1, 2, 3, 7, 11, 19, 43, 67, 163]

def r(x):
    return mp.exp(mp.pi * mp.sqrt(x))

for d in H:
    print(f'r({d}) = {mp.nstr(r(d), 50, min_fixed=-mp.inf)}')
```

The outputs:

```
r(1)   = 23.140692632779269005729086367948547380266106242 6
r(2)   = 85.019695223207217582510872858830968583830749237124
r(3)   = 230.76458831914587924007515393100900168785407806548
r(7)   = 4071.9320952252610985245683325575979166518448579633
r(11)  = 33506.143065592438766681550962819177911973323522262
r(19)  = 885479.77768015431949753789348171962682071428650187
r(43)  = 884736743.99977746603490666193746207858537684739915
r(67)  = 147197952743.99999866245422450682926131257862850819
r(163) = 262537412640768743.99999999999925007259719818568888
```

Looking carefully, $r(163)$ is Ramanujan's constant!

As d increases, $r(x)$ converges towards integers - especially for $x \in \{43, 67, 163\}$. To understand this precise behaviour, we must introduce ourselves to the '*j-invariant*'.

6.1 J-Invariant

The j-invariant $j(\tau)$ is a modular function. Unlike the 2π-periodicity of $\sin(x)$, modular functions are periodic in two directions on the complex plane.

For any $\tau \in \mathbb{C}$ in the upper half-plane:

$$\mathbb{H} = \{z = x + iy \in \mathbb{C} \mid y > 0\}$$

$j(\tau)$ is invariant under the transformations $\tau \to \tau + 1$ and $\tau \to -1/\tau$. Generally, for any integers a, b, c, d where $ad - bc = 1$:

$$j\left(\frac{a\tau + b}{c\tau + d}\right) = j(\tau)$$

Rationalising the input shows that this transformation preserves the structure of $\mathbb{H}$:

$$\mathrm{Im}\left(\frac{a\tau + b}{c\tau + d}\right) = \frac{\mathrm{Im}(\tau)}{|c\tau + d|^2} > 0$$

This symmetry allows $j(\tau)$ to act as a 'bridge' between transcendental and algebraic worlds. Although $j(\tau)$ is a crucial tool for classifying elliptic curves (which we are not concerned about),

we use it here as a valuation tool providing the exact integer values needed to prove why $e^{\pi\sqrt{d}}$ results in a near-integer. Evaluating $j(\tau)$ at a Heegner point - a coordinate tied to $\mathbb{Q}(\sqrt{-d})$ - forces the output to 'fall' into an integer! For $d = 67$:

$$j\left(\frac{1+i\sqrt{67}}{2}\right) = -147,197,952,000$$

6.2 Final Computation

The final link between this integer and our computation lies within the j-invariant's infinite series expansion:

$$j(\tau) = \frac{1}{q} + 744 + 196884q + 21493760q^2 + \dots$$

where $q = e^{2\pi i\tau}$. For our specific τ:

$$q = e^{2\pi i\left(\frac{1+i\sqrt{67}}{2}\right)}$$

$$q = e^{\pi i - \pi\sqrt{67}}$$

Using Euler's identity:

$$q = -e^{-\pi\sqrt{67}}$$

Therefore:

$$j(\tau) \approx -e^{\pi\sqrt{67}} + 744$$

(We ignore the higher terms as q is infinitesimally small). By substituting our known integer value for $j(\tau)$:

$$-147,197,952,000 \approx -e^{\pi\sqrt{67}} + 744$$

$$e^{\pi\sqrt{67}} \approx 147,197,952,000 + 744 = 147,197,952,744$$

This is exactly why our computation for $r(67)$ returned a value ending in .999998...!

7 Conclusion

Initially a memetic placeholder, 67 is revealed as the foundation of algebraic stability. As a Heegner number, it guarantees UF, nurtures Euler's lucky primes, and provides the symmetry required to align transcendental values with integers. 67 proves that even a digital punchline can behold incredible mathematical secrets.

(Six Sevennn!)