Some Complex Maths

Preface

I hope that this essay can provide a more fun approach to *some complex maths* topics. Speaking of the word complex please take this as a warning it will appear a lot.

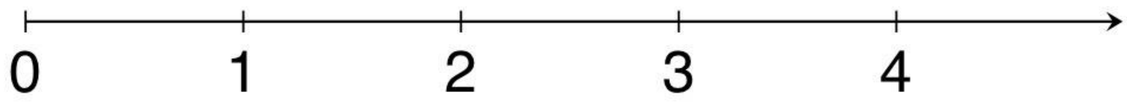
Apologies in advance for any wishy washy notation as this was written on a mobile device, so naturally some notation was difficult to incorporate. Hence, why some of the working out has a different font as it was written on an app called Nuten.

All of that aside, let's get on with *some complex maths*.

Understanding Real numbers

Let's start with the basics before things get too *complex*.

Real numbers are numbers that can have up to an infinite number of decimal places as said by Stevin around 1600. For example, 3 is a real number because 3 has an infinite number of decimal places (if you write 3 as 3.000000….). Another interpretation of a real number is using a number line, continuing with our example using 3:



As you can see 3 can be represented on a number line as shown above.

Understanding Complex numbers

The imaginary unit i is called imaginary because there is no real number where its square root is negative. So we expand the number system to allow for the square root of negative numbers to exist.

Complex numbers, this name is really misleading as complex numbers are actually very simple, they are numbers that are in the form $a+bi$ where a and b are real numbers and $i=\sqrt{-1}$.

Fun fact: it is a common misconception that complex numbers were used to solve quadratic equations when they were first discovered, the historical motivation was actually to solve cubic equations.

Let us consider a cubic of the form $y^3=py+q$ where the image below is the formula that gives us the solutions:

$$x = u - v$$

$$\boxed{= \sqrt[3]{\frac{-q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}} - \sqrt[3]{\frac{q}{2} + \sqrt{\frac{q^2}{4} + \frac{p^3}{27}}}}$$

If the part that is within the square root is negative, then we get no real solutions because we cannot square root a negative.

Before 1572, mathematicians would ignore this idea of complex numbers and only focus on real solutions to equations. This all changed when; Bombelli decided to look at these cubics and say to himself "what if we pretended √-1 existed so we could solve these equations". Effectively treating √-1 as its own element.

We can find this solution using the cubic formula substituting in p=15, q=4 :

$$\sqrt[3]{2+11i} - \sqrt[3]{-2-11i} = x$$

Since $(2+i)^3=2+11i$ and $(-2-i)^3=-2-11i$

$\therefore$ x= 2+i -(-2-i)

The fancy 3 dots are just a cool way to say therefore

x=4,

This solution is only obtainable if we consider complex numbers which makes them essential.

When working with complex numbers we need to work with i like we would a surd (think of a surd as keeping something as a square root). We can do this because, when thinking logically about how $i=\sqrt{-1}$ it shows that i is fundamentally just a surd.

When it comes to doing arithmetic, consider the cases for √2 and i

Addition:

$$\sqrt{2} + \sqrt{2} = 2\sqrt{2}$$

i+i=2i

This is similar to how in algebra:

y+y=2y

Subtraction:

$$2\sqrt{2} - \sqrt{2} = \sqrt{2}$$

2i-i=i

Multiplication:

$$\sqrt{2} \times \sqrt{2} = 2$$

ixi=-1

because (√-1x√-1=ixi)

Division:

Realising a denominator is the same process as rationalising a denominator.

This is the case for rationalising a denominator:

$$\frac{2}{1 + \sqrt{2}}$$

Multiply by the conjugate (just means change the sign in the middle to the opposite sign) on top and bottom to get a difference of 2 squares to counteract the effect of the square root as we will be left with only squared terms.

$$\frac{2(1-\sqrt{2})}{(1+\sqrt{2})(1-\sqrt{2})}$$

Expand brackets and simplifying:

$$\frac{2-2\sqrt{2}}{(1+\sqrt{2})(1-\sqrt{2})}$$

$$\frac{2-2\sqrt{2}}{1-\sqrt{2}+\sqrt{2}-(\sqrt{2})^2}$$

$$\frac{2-2\sqrt{2}}{1-(\sqrt{2})^2}$$

$$\frac{2-2\sqrt{2}}{1-2}$$

$$\frac{2-2\sqrt{2}}{-1}$$

$$2\sqrt{2}-2$$

Which gets us our final answer!

For the case using i:

$$\frac{2}{1+i}$$

Let’s start off just as we did in the last example, by multiplying top and bottom by the conjugate

Fun fact: here we call it the complex conjugate to be more fancy.

$$\frac{2(1-i)}{(1+i)(1-i)}$$

$$\frac{2-2i}{1^2-i^2}$$

$$\frac{2-2i}{1--1}$$

Note that subtracting a negative is the same as adding

$$\frac{2-2i}{2}$$

$$1-i$$

We have reached our final answer. :)

Groups

A set is just a collection of items in maths, for example the set of numbers from 1 to 10 is just {1,2,3,4,5,6,7,8,9,10}.

A group may sound very basic ,but again this is misleading as groups are actually quite *complex*. More *complex* than complex numbers believe it or not. A group is a set that satisfies a specific criteria because they are picky. This criteria includes the following:

1. A binary operation-just a fancy way of saying a rule that maps 2 or more elements onto a new element. For example, addition could be a rule.
2. Associativity-$(a+b)+c = a+(b+c)$ is true or in simple terms brackets don't matter.
3. Inverse- the operation that has the opposite effect to the binary operator, so for example, addition's inverse is subtraction.
4. Identity- an element that has no effect when applying the binary operator for example, you can add 0 to something with no effect 0 is the additive inverse.
5. Closure- you can apply the binary operator without needing elements outside the group for example, 1+1=2 since 2 is real the additive set of real numbers is closed.

It is important to note that since the computational behavior of complex numbers is very similar to surds with real numbers; the set of the complexes can satisfy the criteria for a group.

Another fun fact is that all of the real numbers are technically complex numbers as they can be written in the form $a+b\mathit{i}$, where $b=0$, and a is real. Therefore all the real numbers are in the set of complexes.

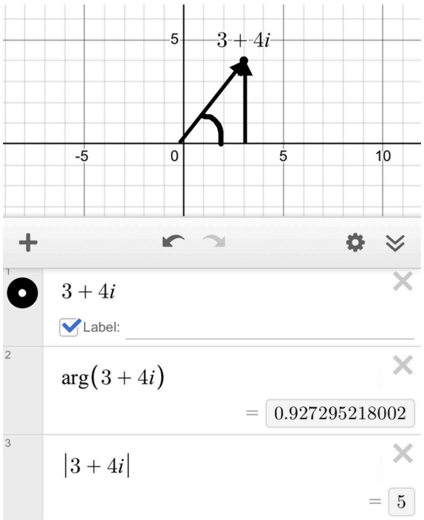
Representing complex numbers

We represent complex numbers as a point on a 2-d plane with 1-line for the real part and 1-line for the imaginary because they have 2 components. Therefore, we can represent complex numbers as a point relative to the origin (0,0). Using this idea we can write complex numbers in the form $r(\cos(\theta)+\mathit{i}\sin(\theta))$ where r is the modulus (distance from the origin) and θ is the angle between the origin and the real line.

We need to consider this as a right-angled triangle and then using what we know about trigonometry to find the modulus and the arg:

1. Since any complex number can be written as $a+b\mathit{i}$ we can use Pythagoras' Theorem to derive that the modulus is $\sqrt{(a^2+b^2)}$.
2. To find the θ we need to take arctan(b/a), we like to use radians here.

For example,



the modulus here is 5 as:

$$\sqrt{3^2 + 4^2} = 5$$

The arg is 0.927 radians as:

$$\arctan\left(\frac{4}{3}\right) = 0.927$$

When multiplying 2 complex numbers in this form we multiply the moduli and add the angles.

Here is how we can derive this:

Imagine we have 2 complex numbers:

$$z_1 = r_1(\cos(\theta_1) + i\sin(\theta_1))$$

$$z_2 = r_2(\cos(\theta_2) + i\sin(\theta_2))$$

$$(z_1)(z_2)=r_1(\cos(\theta_1)+i\sin(\theta_1))(r_2(\cos(\theta_2)+i\sin(\theta_2))$$

Group the r_1 and r_2 together to get:

$$(z_1)(z_2)=r_1r_2(\cos(\theta_1)+i\sin(\theta_1)(\cos(\theta_2)+i\sin(\theta_2))$$

Expanding the brackets to get

$$(z_1)(z_2)=r_1r_2(\cos(\theta_1)\cos(\theta_2)+i\cos(\theta_1)\sin(\theta_2)+i\sin(\theta_1)\cos(\theta_2)+i^2\sin(\theta_1)\sin(\theta_2))$$

Note that the set of complex numbers is commutative when it comes to multiplication and addition so the order things are written won't affect the final result.

$$(z_1)(z_2)=r_1r_2(\cos(\theta_1)\cos(\theta_2)+i\cos(\theta_1)\sin(\theta_2)+i\sin(\theta_1)\cos(\theta)-\sin(\theta_1)\sin(\theta_2))$$

$$(z_1)(z_2)=r_1r_2(\cos(\theta_1)\cos(\theta_2)-\sin(\theta_1)\sin(\theta_2)+i(\sin(\theta_1)\cos(\theta_2)+\cos(\theta_1)\sin(\theta_2))$$

Using the addition formulae which I will give to you now

$$\sin(\theta_1\pm\theta_2)=\sin(\theta_1)\cos(\theta_2)\pm\sin(\theta_2)\cos(\theta_1)$$

$$\cos(\theta_1\pm\theta_2)=\cos(\theta_1)\cos(\theta_2)\mp\sin(\theta_1)\sin(\theta_2)$$

Substituting these in we get:

$$(z_1)(z_2)=r_1r_2(\cos(\theta_1+\theta_2)+i\sin(\theta_1+\theta_2))$$

:) a smiley face for a happy result I'd say.

As you can imagine, when dividing complex numbers, you can do the opposite of what we did when we multiplied them i.e. dividing the moduli and subtracting the arguments. This can be proved through the same process but I won't bore you with that.

<u>De Moivre's theorem</u>

$z^n= r^n(\cos(n\theta)+i\sin(n\theta))$ this is called De Moivre's theorem discovered in 1707.

To prove this we need to understand a technique called proof by induction. This can be visualised by a simple set of dominos. Effectively what proof by induction is saying is that, if the first domino can be knocked down and any kth domino and the domino right after the kth domino (the (k+1)th domino) then you can knock down all the dominos. Now linking this back to numbers we can say proof by induction is used to prove statements true for all positive integers (consider the positions of the dominos as each integer and knocking it down means the statement is true)

Let's now have a go at proving the great De Moivre's theorem which is shown below.

$$r^n(\cos(\theta)+i\sin(\theta))^n = r^n(\cos(n\theta)+i\sin(n\theta))$$

For n=1 (first domino):

$$r^1(\cos(\theta)+i\sin(\theta))^1 = r(\cos(\theta)+i\sin(\theta))$$

$$r^1(\cos(1\theta)+i\sin(1\theta)) = r(\cos(\theta)+i\sin(\theta))$$

So we say LHS=RHS which just means left hand side equals right hand side

So it is true for n=1

Now let's assume true for n=k, where k is a positive integer:

$$r^k(\cos(\theta)+i\sin(\theta))^k=r^k(\cos(k\theta)+i\sin(k\theta))$$

Now let us consider the case where n=k+1:

$$r^{k+1}(\cos(\theta)+i\sin(\theta))^{k+1}=r^k(\cos(\theta)+i\sin(\theta))^k r(\cos(\theta)+i\sin(\theta))$$

Think of splitting the exponent like this as the inverse of adding the exponents when multiplying with the same base

We substitute our assumption in to get:

$$r^{k+1}(\cos(\theta)+i\sin(\theta))^{k+1}=r^{k+1}(\cos(k\theta)+i\sin(k\theta))(\cos(\theta)+i\sin(\theta))$$

Expanding the brackets to get:

$$r^{k+1}(\cos(\theta)+i\sin(\theta))^{k+1}=r^{k+1}(\cos(k\theta)\cos(\theta)+i\cos(k\theta)\sin(\theta)+i\sin(k\theta)\cos(\theta)+i^2\sin(k\theta)\cos(\theta))$$

Group the real and imaginary elements:

$$r^{k+1}(\cos(k\theta)\cos(\theta)-\sin(k\theta)\cos(\theta))+i(\sin(k\theta)\cos(\theta)+\cos(k\theta)\sin(\theta))$$

Using the addition formulae:

$r^{k+1}(\cos(\theta)+i\sin(\theta))^{k+1}=r^{k+1}(\cos(k\theta+\theta)+i(\sin(k\theta+\theta))$

$r^{k+1}(\cos(\theta)+i\sin(\theta))^{k+1}=r^{k+1}(\cos(\theta(k+1)))+i(\sin(\theta(k+1)))$

Therefore, De Moivre's theorem holds for n=k+1

Hence, we have proven by mathematical induction that De Moivre's theorem holds for all positive integers.

Euler's Identity

The most wonderful form of representing complex numbers is $re^{i\theta}=r(\cos(\theta)+i\sin(\theta))$.

Let's try and prove this,

Taylor Expansions are a way we can write functions as polynomials.

The Taylor Expansions we will be using are:

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} \ldots.$$

$$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} \ldots.$$

$$\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} \ldots.$$

starting off with the Taylor Series for e^x

Substituting ix for x to get

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} \ldots.$$

Since, $i^2=-1$, $i^3=-i$, $i^4=1$ and $i^5=i$ …

$$e^{ix}=1+ix-\frac{x^2}{2!}-\frac{ix^3}{3!}+\frac{x^4}{4!}\ ....$$

Separate real and imaginary terms:

$$e^{ix}=\left(1-\frac{x^2}{2!}+\frac{x^4}{4!}\ ....\right)+i\left(x-\frac{x^3}{3!}+\frac{x^5}{5!}\ ....\right)$$

Substituting the Taylor Expansions of cos(x) and sin(x) in:

$$e^{ix}=\cos(x)+i\sin(x)$$

Multiplying by r on both sides to include the modulus:

$$re^{ix}=r(\cos(x)+i\sin(x))$$

Which gets us our final result.

:)

Hopefully you found that helpful and it made a very *complex* topic feel more simple. As well as widening your perspective on how complex maths is.

References

1. https://youtu.be/5Ui5W_BDa8s?si=y3J8kbHg1tee8RJZ this was useful for some of the history facts
2. Nuten was used to write some of the working out.
3. The complex plane came from https://www.desmos.com/calculator/n1698xdz63
4. the number line was made in https://themathworksheetsite.com/numline.html
5. The image for the cubic formula is from https://medium.com/puzzle-sphere/the-cubic-formula-the-quadratic-formulas-big-brother-3fcb16759ce