<u>The Story of Geometry-</u>

<u>From a Flat World to Curved Shapes:</u>

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Around 300BC, Euclid outlined what would become the standard of geometry that everything was based around, from navigation at sea to architecture and engineering. It was seen as an idea that nobody could doubt or question, it was thought these few basic assumptions could describe the world- and everything- perfectly. However, there was one, that stood alone- less obvious and not as definite as the others… **[1]**

<u>Euclidean geometry was elegantly summarised into five postulates:</u>

1. A straight line can be drawn between any two points

2. A finite straight line can be extended indefinitely

3. A circle can be drawn with any centre and radius

4. All right angles are equal

5. The Parallel Postulate: Through any point not on a given line, exactly only one parallel line can be drawn. *This is now known as Playfair's Axiom because in 1795 John Playfair did work on the fifth postulate.*

All the postulates were the foundation blocks of geometry: there was no need for proof. However, the parallel postulate was less intuitively obvious, and no mathematician could form a proof for it for centuries to come. It was so important for mathematicians to understand, because the answer to this question would change how we understood planes, lines, shapes and all of geometry. Is Euclid's Fifth Postulate true or is there something else waiting to be discovered?

One mathematician interested in trying to prove the infamous fifth postulate was Giovanni Girolamo Saccheri, most famous for his work around Euclid's postulates. He attempted to prove Euclid, using a proof by contradiction. **[2]**

Figure 1

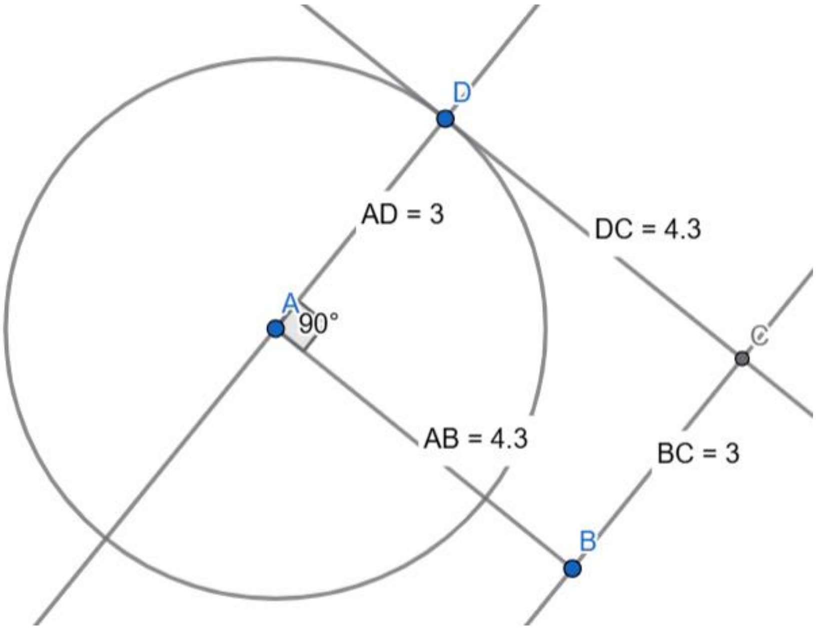


Figure 1 shows Saccheri's quadrilateral. He constructed a quadrilateral ABCD, where AB was a straight line with two equal ended perpendiculars: AD and BC. An important note was that the summit angles (the two top angles) were always equal. This was because both lines are perpendicular to the base AB so if the base angles are the same, the summit angles must be the same as well due to symmetry.

For the summit angles, there were three hypotheses: the right-angle hypothesis (Euclidean geometry where there was 1 parallel through any given point), the obtuse angle hypothesis (spherical geometry where there are no parallels through any given point) or the acute angle hypothesis (hyperbolic geometry where there is more than one parallel through any given point). Saccheri originally set out to attack alternatives to Euclid, but the acute angle hypothesis derived a result that worked and was not a contradiction. Despite Saccheri's near success of discovering hyperbolic geometry, mathematicians remained unconvinced with his findings. It took almost another century for the next stage of our story to unfold... **[3]**

In the early 1820s, János Bolyai started investigating the problem with the parallel postulate and became deeply absorbed into it. His father Farkas Bolyai wrote a famous letter to him, warning him... "I have traversed this bottomless night, which extinguished all light and joy of my life. I entreat you, leave the science of parallels alone... Learn from my example." Yet János did not heed his father, and unlike others, he wondered whether the fifth postulate could be proven without using the other 4. What if it was completely independent? János imagined an image where instead of one parallel line through a point, there could be more than one line. He did this by considering space as curved.

As described by Euclid, a straight line is merely the shortest route between two points. If these two points are on a curved surface, there can be infinitely many parallel lines. They may not look straight but bent, but because the space was curved, these lines would still be straight. He completely changed the idea of a straight line is. In hyperbolic space. What we see as a curved line are just geodesics in a curved plane. A geodesic is simply the shortest path between two points on a curved surface. (*See Figure 2)*

Figure 2

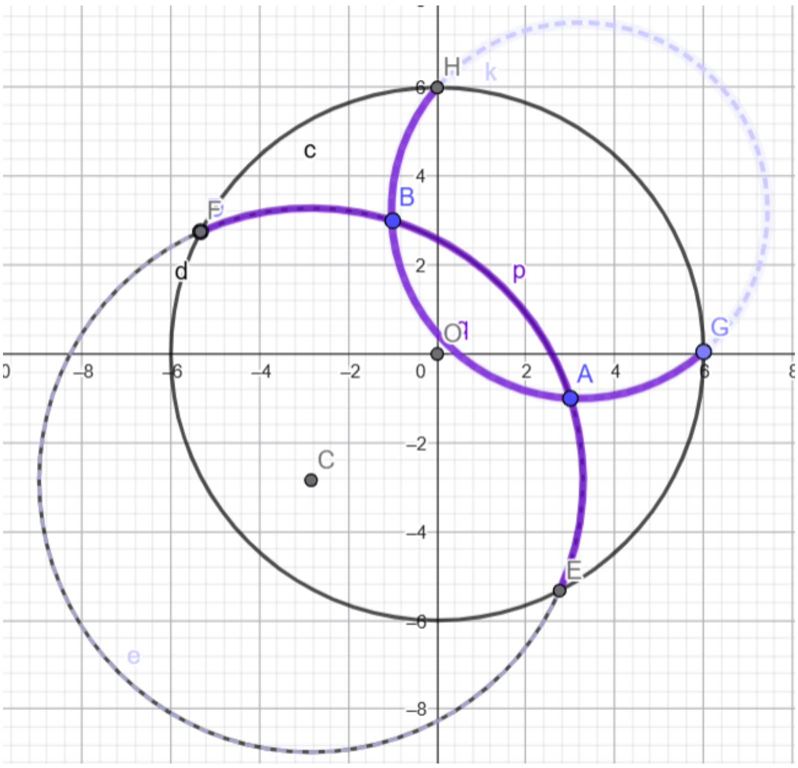


Figure 2 shows the Poincaré Disk Model, where the black circle centred at the origin represents the hyperbolic plane. The purple arcs are geodesics in the plane. A geodesic is the shortest path between two points on a curved surface. It is important to note, that the actual model has a plane with radius 1, and points on the boundary represent points infinite distance away. It is important to note that although the plane seems to be a flat two-dimensional plane, it is in fact a curved plane.

In Euclidean geometry, we can easily take the distance between two points by taking the square root of the difference of the x values squared plus the difference of the y values squared. But in the Poincaré disk model, we cannot do this because the whole infinite plane is represented by the circle, so we cannot measure distance between two points by normal means as points on the edges are infinitely far away from each other. So, we need a different way to find distance in the Poincaré disk model.

To do this we can use the cross-ratio. Say we have two points P and Q inside the plane. We can draw the geodesic between them and extend it indefinitely to the boundary. We know we can do this because of Euclid's second postulate which says that any finite line can be extended indefinitely. Where the line intersects the boundary, we form points A and B. Because we cannot measure distance directly, the cross ratio compares the positions of the 4 points relative to each other. One important point about the cross ratio is that it remains the same under transformations of the disk because we are not using their points on the plane, just their positions relative to each other. These transformations are known as Möbius transformations, where lines and circles are mapped onto other lines and circles. It preserves angles, and hyperbolic geometry, as cross-ratio is unchanged this means hyperbolic distance stays the same.

Cross-ratio: [P, Q, A, B] = |PB|*|QA|/|PA|*|QB|

But why are the edges of the hyperbolic plane (represented as a circle) infinitely far away? In Euclidean geometry, a circle grows in area by πr^2, but in hyperbolic geometry the area grows exponentially with its radius. Therefore, as the radius increases, the area of the circle approaches infinity. If A, B, P and Q are all real points, the Poincaré length of P to Q is defined as: **[4]**

d(A, B) = |ln([P, Q, A, B])|

But what happens to triangles in hyperbolic planes is incredible. Let's consider how the angle sum changes. Let us call three angles of a triangle A, B and C. In Euclidean geometry, A+B+C = π, but in hyperbolic, A+B+C < π. There is always an angle defect in a triangle in the hyperbolic plane. This is because the hyperbolic plane has a negative curvature and everything curves inwards, so angles become smaller and therefore the result become less than π.

Angle defect = π – (A+B+C)

A fascinating result in hyperbolic geometry is that if two triangles are similar, they are congruent. This is because of the Gauss – Bonnet Rule which states that **Area = Angle defect = π – (A+B+C).** Now this tells us that the area in hyperbolic geometry is directly related to the angles, not the lengths of the triangle.

Another aspect of hyperbolic geometry is that it is so intimately linked to hyperbolic functions. The Euclidean distance between the origin (O) and any point F within the Poincaré Disk model is defined as |z|.

In hyperbolic geometry, **d(F, O) = ln(1+|z|/1- |z|) = 2artanh(|z|)!**

We have |z|<1 as the disk has radius 1, and a point F lies inside the disk.

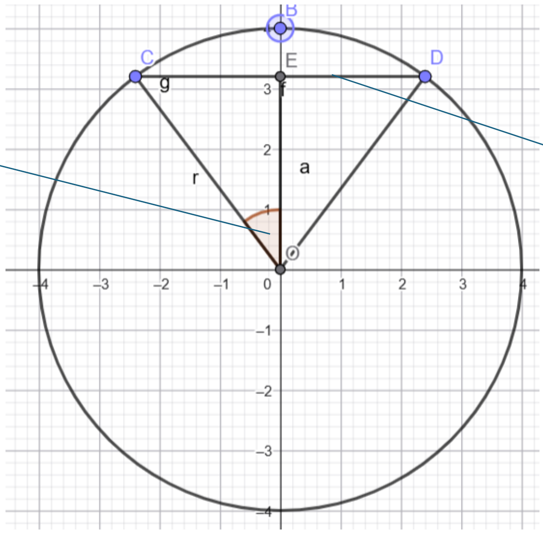
The result we get for this distance is exactly the same as 2 times the inverse of tanh!

We see a direct relation between hyperbolic functions and the Poincaré Disk model!

(See below for Figure 3, where using it we will prove an interesting result that further demonstrates the link with hyperbolic functions)

Figure 3

This angle is **2π/n**



This length is

p/2n

Figure 3 shows the circle which represents the entire Hyperbolic Plane. The circle is the plane of the Poincaré Disk model, and this is a way to find the perimeter of it. Let's take an n sided polygon, as n approaches infinity, the polygon will become closer in shape to a circle. So, we can therefore split the polygon into triangles and work from there.

[Proof] Let's call this polygon P(n). Join the origin to each vertex to split P(n) into n equal triangles. Let's focus on $\triangle OCD$. Drop a perpendicular from O to the side CD and call this point E. Point E bisects CD. **[5]**

From the hyperbolic sin rule which states that:
sinh(a)/sin(A) = sinh(b)/sin(B).

We substitute our values into the rule from *Figure 3,* to get sinh(p/2n)/sin(π/n) = sinh(r)/1.

We can rearrange the equation into a more useful form.

sin(π/n) = sinh(p/2n)/sinh(r)

Now we can consider what will happen as n approaches infinity. If we have a function sin(x), for very small x we can approximate sin(x) ≈ x. Therefore here, sin(π/n) is approximately equal to π/n. This is also true for a function sinh(x) ≈ x , so sinh(p/2n) is approximately equal to p/2n. These approximations come from the first term of their respective Taylor Series expansions.

∴ We can substitute these into our equation and we get: (π/n)* sinh(r) = p/2n

Rearranging, p = 2πsinh(r)

An important point is that here we refer to r as the hyperbolic distance between the centre and a point on the circumference of a hyperbolic circle within the plane, and as r increases the circle approaches the boundary of the hyperbolic plane but can never actually reach it.

Figure 4

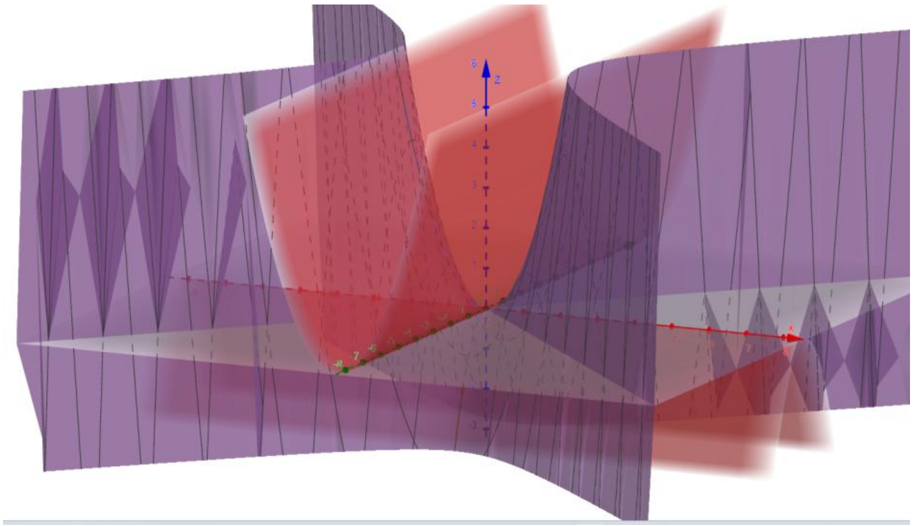


Figure 4 shows a hyperbolic paraboloid. It is also known as a saddle surface. To be able to clearly see the hyperbolic plane, I constructed z = x^(2)-y^(2). These red planes represent z = x^(2) and z = y^(2). ***[6]***

We need to consider the cross sections to get a better grasp of the surface. The cross sections are the red curved planes. Along perpendicular lines, the cross sections are in opposite directions and also are facing upwards and the other one is downwards. This is a key property as it shows how space in the hyperbolic world is negatively curved. This is why straight lines (geodesics) are different than In Euclidean geometry, as multiple parallel lines can pass through a single point because of the negative curvature, because geodesics diverge and therefore there are infinitely many lines that are parallel through a single point.

The discovery and understanding of hyperbolic geometry completely reinvented geometry as a whole and had vital importance in many fields from physics to graphics and computer science. It helped to shape mathematics with important concepts like hyperbolic functions which has applications in modern engineering, for example in building bridges with hanging cables. They can additionally arise in topics like heat and wave equations, as they often include exponential growth or decay, which relate to the hyperbolic definitions of sinh(A) and cosh(A). Furthermore, Einstein's theory of general relativity was only possible due to the discovery of curved space. Einstein proposed that gravity is not a force that pulls on objects, instead it bends the fabric of spacetime itself. Let's take the sun which bends the spacetime around it and then the Earth orbits the Sun as it is moving in the shortest possible line (a geodesic).

It opened up countless new ideas that until previously, even the most brilliant of minds never imagined. The breakthrough in hyperbolic geometry led many mathematicians to wonder whether a new type of geometry was possible. Going back to Saccheri's quadilateral, what if the summit angles were obtuse instead of acute? This would eventually lead to a new discovery, in elliptical geometry, but I'll leave that for next time.

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