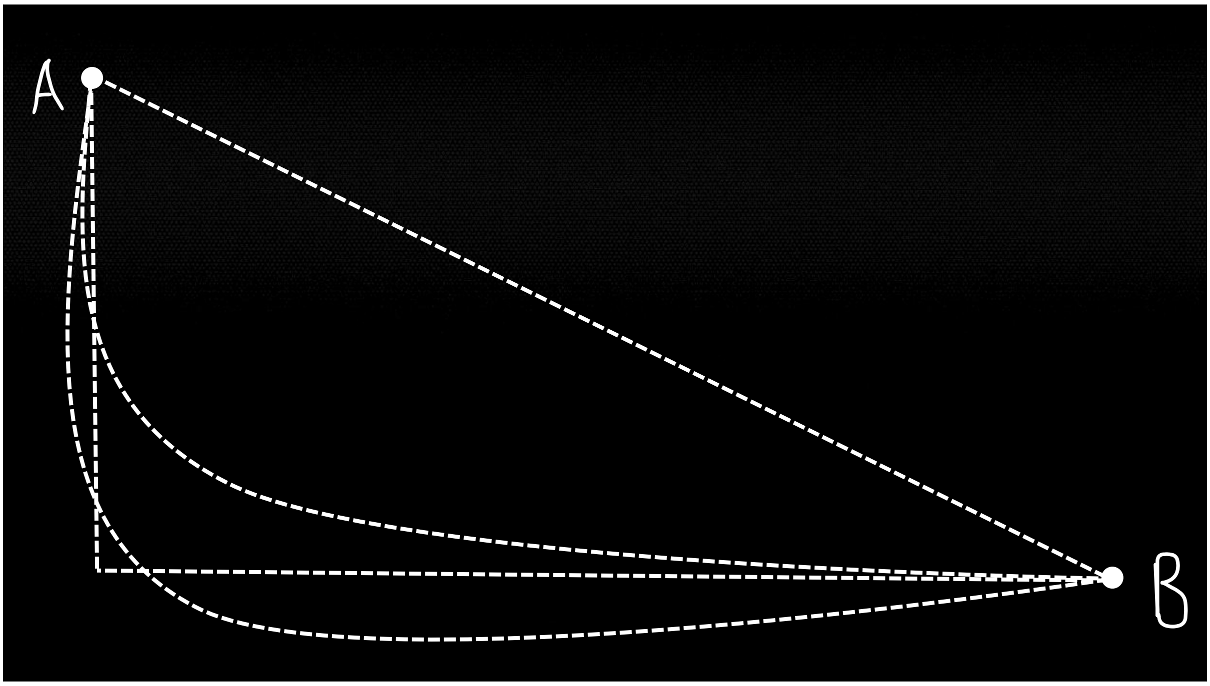
The Brachistochrone Curve: A Whistlestop Tour from A to B

By Tanish Dhir

1. Introduction

The somewhat intimidating word *brachistochrone* comes from the Greek *brachistos*, meaning shortest, and *chronos*, meaning time. So the brachistochrone is, literally, the curve of shortest time - and the question it answers is this: if you place a bead at a point A and let it slide due to gravity and without friction, down a wire to a lower point B, what shape should the wire take so that the bead arrives as quickly as possible?



So many paths… but which one is best?

Your first instinct might be to try a straight line. It is, after all, the shortest *distance* between the two points. However, the shortest distance and the shortest time are not the same thing. A wire that plunges steeply downward at the start lets the bead build up speed much faster; that extra speed more than compensates for the longer path. The straight line is not the answer; figuring out what the answer is? Far from obvious.

The problem was posed publicly in 1696 by Johann Bernoulli, a Swiss mathematician. Johann Bernoulli was a brilliant mathematician - contentious and not above a little showmanship, he issued the brachistochrone as an open challenge to the mathematicians of Europe, implying that most of them would fail. Five correct solutions arrived within a year, from Newton, Leibniz, l'Hôpital, Jakob Bernoulli, and Johann himself. Newton reportedly solved it overnight after a long day at the Royal Mint and submitted his answer anonymously. Johann, upon receiving it, is said to have remarked: *"I recognise the lion by his claw."*

Now, on the face of it, this looks like the kind of problem that would require some serious machinery; we are trying to minimise something over all possible curves, which sounds like it should involve heavy-duty calculus. But here is the good news: it doesn't! The solution we are going to attain utilises nothing more than similar triangles, circle theorems and a beautiful analogy with the behaviour of light.

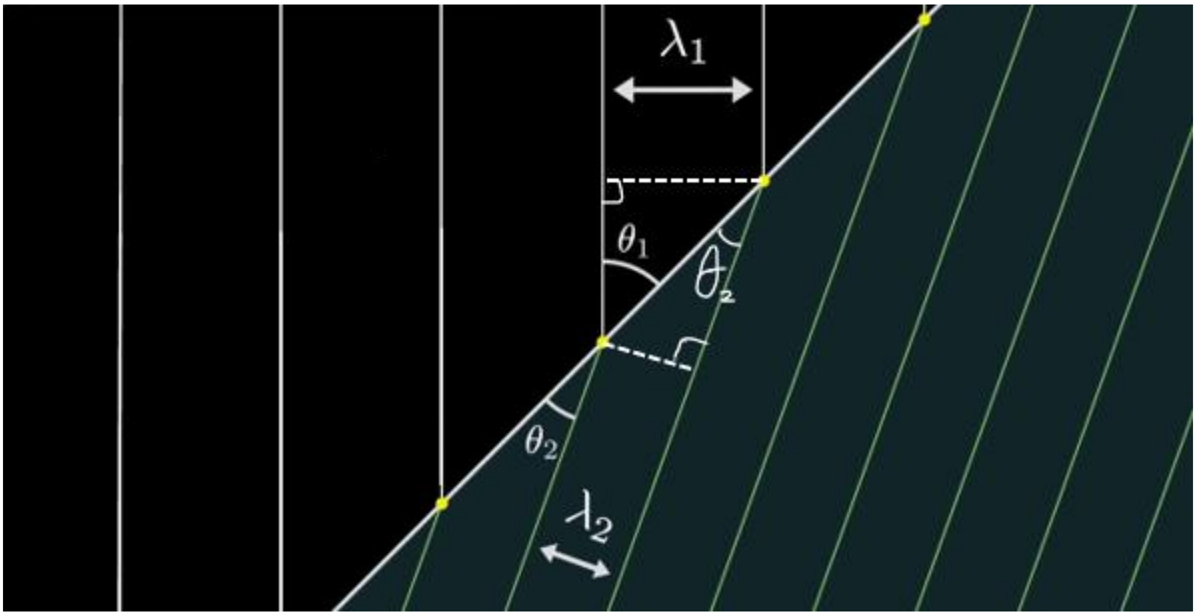
The solution has three steps. First, we state Fermat's Principle and prove Snell's Law, the rule that governs how light bends when it crosses from one medium into another, directly from the geometry of wavefronts. Second, we meet the cycloid and prove a key property it satisfies, using a clever geometric argument. Third, we show that any path of fastest descent must satisfy that same property, which forces the cycloid upon us as the unique answer. And then we are done!

2. Fermat's Principle and Snell's Law

By Fermat's Principle, light travels between two points along the path that takes the least time. This can be intuitively understood by considering how light behaves when passing between media of different speeds: rather than following the shortest distance, it bends to optimise travel time, spending more distance in the faster medium. At a deeper level, this arises from the wave nature of light, where many possible paths contribute, but only the path of least time is reinforced while others destructively interfere, making it the path that is physically observed.

It is therefore reasonable to model the bead as a ray of light, with gravity replaced by a medium that becomes less optically dense with depth.

When a ray of light passes from one medium into another, say from air into glass, it bends. This is refraction and the precise rule governing the bending is called Snell's Law. We are going to prove it from scratch, because we will need it shortly in a rather surprising context. The proof uses the idea of a wavefront. Rather than thinking of light as a single ray, think of it as a wave: a whole family of parallel lines sweeping through space, all moving in the same direction, equally spaced. These parallel lines are the wavefronts: surfaces of constant phase. The ray is just the direction perpendicular to them: the direction in which the wave is travelling.



A wavefront diagram of a wave passing from one medium into another.

Consider two adjacent wavefronts as they cross the boundary between the two media. In medium 1, successive wavefronts are separated by wavelength λ_1; in medium 2, by λ_2. Call L the length of the segment of the boundary between the two points where one wavefront meets it (the two yellow dots in the diagram). By basic trigonometry, this length can be expressed in terms of either medium:

$$L = \frac{\lambda_1}{\sin\theta_1} = \frac{\lambda_2}{\sin\theta_2}$$

Reciprocating both sides gives:

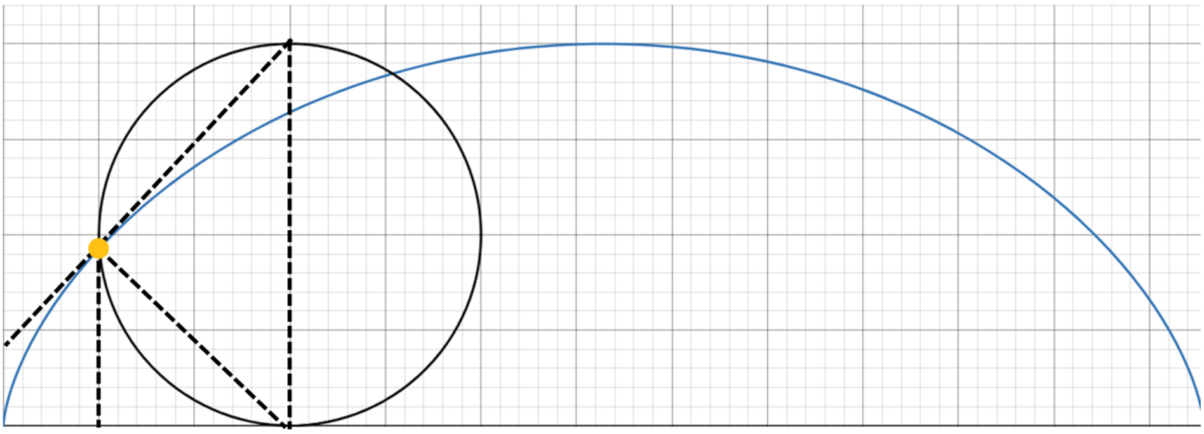
$$\frac{\sin\theta_1}{\lambda_1} = \frac{\sin\theta_2}{\lambda_2}$$

Now we will use the wave relation $c = f\lambda$. The frequency f is the same in both media (it is set by the source and does not change at the boundary), so the wavelength is proportional to the wave speed: $\lambda \propto v$. Replacing λ_1 with v_1 and λ_2 with v_2 gives Snell's Law in the form we will use:

$$\frac{\sin\theta_1}{v_1} = \frac{\sin\theta_2}{v_2}$$

3. The Cycloid and a Clever Geometric Proof

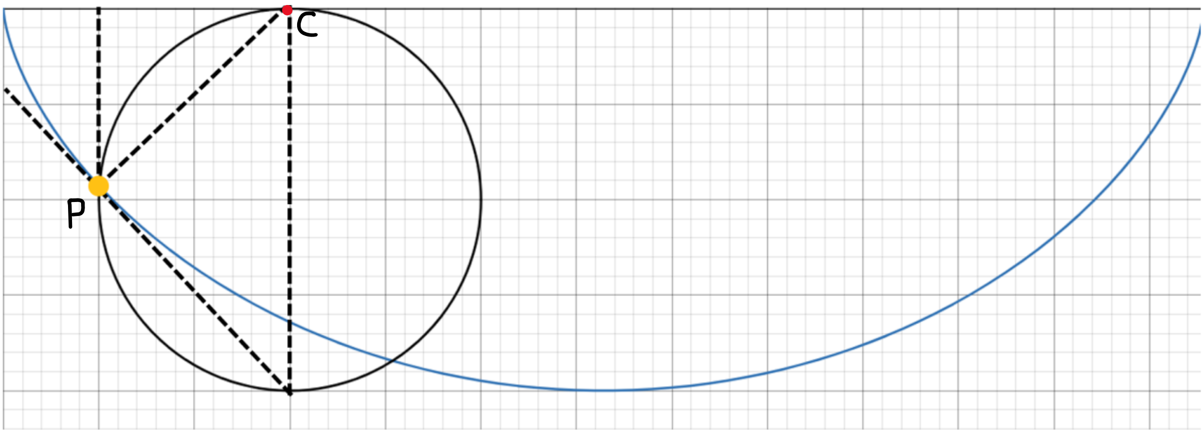
Before we begin, you can play around with the cycloid using [this Desmos link].



Before we can say the cycloid is the answer, we need to know what a cycloid is. Take a circle and roll it along a flat surface without slipping. Fix your eye on a single point (the yellow dot on the diagram above) on the rim of the circle and watch where it goes. The curve it traces out is the cycloid - a smooth arch that starts at a sharp cusp, rises, falls and comes back to another cusp each time the marked point touches the ground again.

What we need to prove is that at every point on the cycloid, the angle θ that the tangent makes with the vertical satisfies:

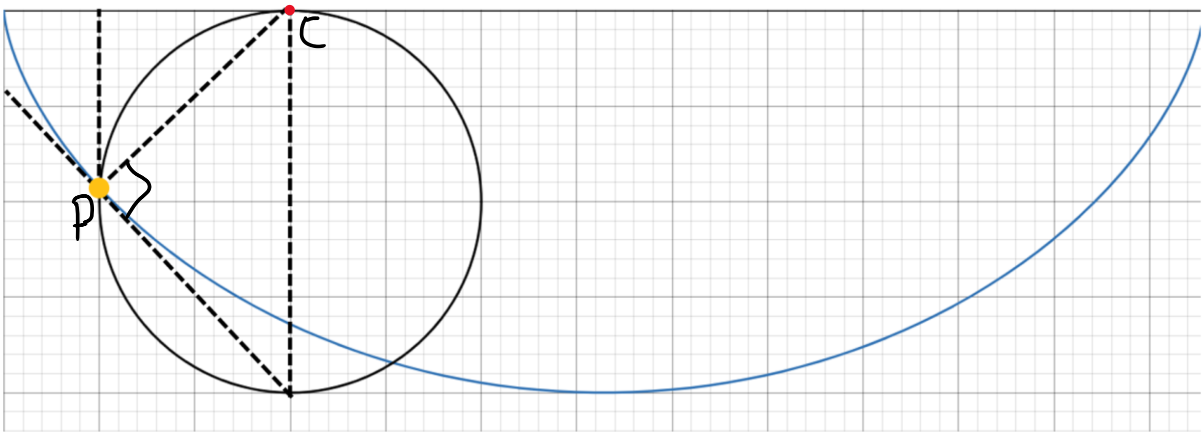
$$\frac{\sin\theta}{\sqrt{y}} = constant$$



Here is the argument. Instead of a wheel rolling on the floor, picture a wheel rolling on the *ceiling* (as pictured above), with the tracing point P hanging below it. As the wheel rolls, P traces out the same shape of cycloid, just oriented differently; all the geometry is identical. Let C be the point where the wheel currently touches the ceiling.

The first key observation is that C acts as the *instantaneous centre of rotation* for the point P. At any given moment, the wheel is pivoting about its contact point with the ceiling – it is as if, for that instant, P is swinging on the end of a pendulum attached to C. This means the velocity of P is perpendicular to the line CP. But velocity is tangent to the path, so:

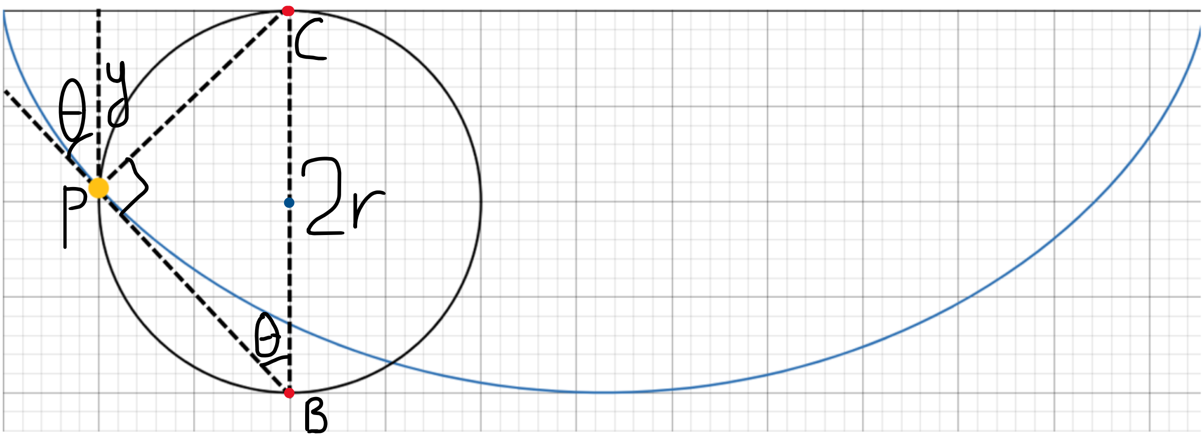
The tangent to the cycloid at P is always perpendicular to the line CP.



Now we have a right angle at P inside the circle (since CP is a radius and the tangent is perpendicular to it). Recall Thales' theorem: any right angle inscribed in a circle must have the diameter as its hypotenuse. Therefore, the line from P - extended at a right angle to CP - must pass through the *diametrically opposite* point from C, which is the bottom of the circle, call it B.

In other words:

The tangent line at P always passes through the bottom of the circle, B.



Now let θ be the angle this tangent line makes with the vertical. We want to find $\frac{sin\theta}{\sqrt{y}}$, where y is the distance from P down to the ceiling (i.e. the depth of P below the rolling surface).

Look at the triangle formed by C, P and B. The line CB is a diameter of the circle, so $|CB| = 2r$. The angle at P is a right angle (by Thales). The angle at B between the vertical and the line BP is θ (since BP is the tangent line and the vertical through B is parallel to the vertical through the diagram). So in the right triangle CPB:

$$|CP| = |CB| \cdot sin\theta = 2r\, sin\theta$$

Now consider the vertical distance y from the ceiling down to P. Since CP ⊥ BP and the tangent make angle θ with the vertical, the line CP makes angle θ with the horizontal. The vertical component of CP is therefore $|CP| \cdot sin\theta$, which is exactly y. Substituting $|CP| = 2r\, sin\theta$:

$$y = |CP| \cdot sin\theta = 2r\, sin\theta \cdot sin\theta = 2r\, sin^2\theta$$

Rearranging:

$$\frac{sin^2\theta}{y} = \frac{1}{2r}$$

$$\frac{sin\theta}{\sqrt{y}} = \frac{1}{\sqrt{2r}}$$

Since r is just the radius of the wheel, which stays constant as it rolls, the right-hand side is a constant. So $\frac{sin\theta}{\sqrt{y}}$ is indeed the same at every single point on the cycloid. That is the result we needed, and it came entirely from similar triangles and Thales' theorem: no calculus whatsoever.

4. Putting It All Together

We now have both halves of the puzzle. From Section 2, Snell's Law tells us that any ray of light travelling through a medium where the speed varies continuously will move so that $\frac{sin\theta}{v}$ stays constant along its path. From Section 3, the cycloid is the curve where $\frac{sin\theta}{\sqrt{y}}$ stays constant.

All we need to do is connect the two!

Here is the remarkably simple connection. By conservation of energy, a bead released from rest and allowed to fall a vertical distance y under gravity has speed:

$$\frac{1}{2}mv^2 = mgy \Rightarrow v = \sqrt{2gy}$$

So $v \propto \sqrt{y}$. The speed of the bead at depth *y* is proportional to $\sqrt{y}$.

Now imagine, as Bernoulli did, that the bead is not a bead at all but a ray of light - and that the plane through which it travels is filled with a fictional optical medium where the speed of light at height *y* is exactly $v = \sqrt{2gy}$.

To spell that out a bit more, imagine the ray passing through many thin horizontal layers, each with a slightly different speed. As we move downward, the media becomes progressively less optically dense, so the light travels faster and bends continuously rather than in sharp jumps.

In this medium, a ray of light obeys Snell's Law, which means it travels along a path where:

$$\frac{sin\theta}{v} = \frac{sin\theta}{\sqrt{2gy}} = constant$$

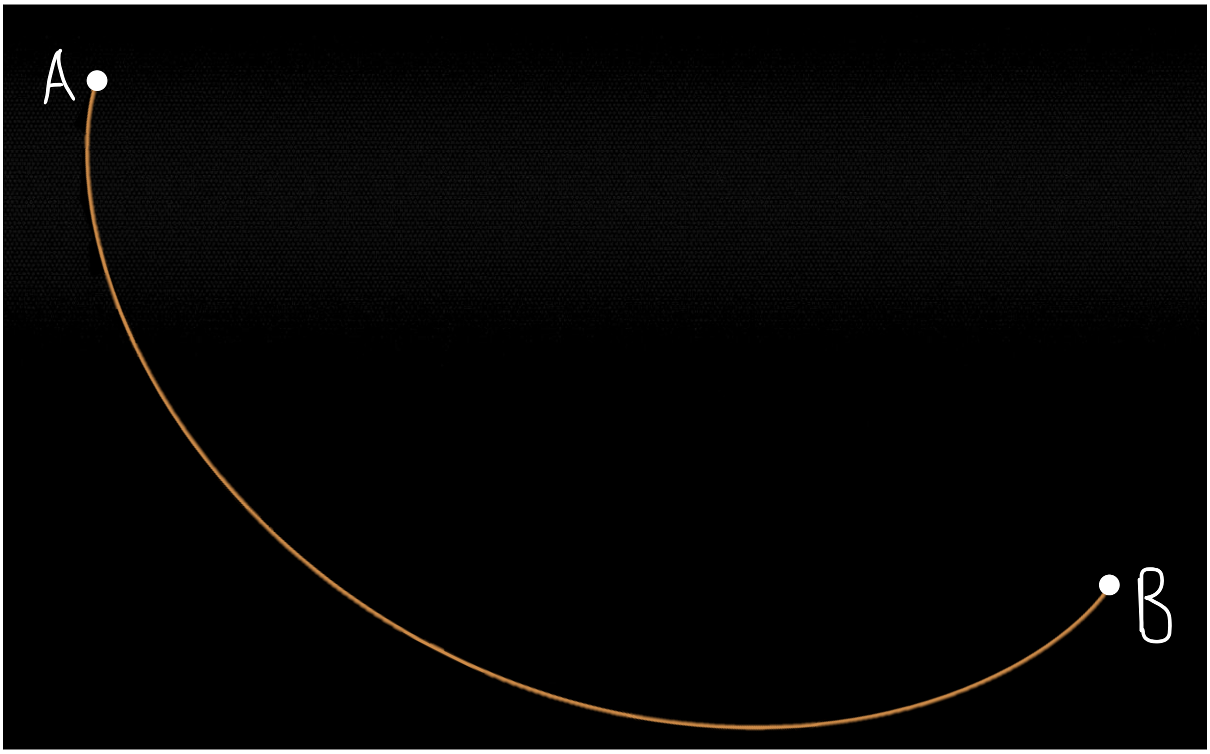
This is the same as saying:

$$\frac{sin\theta}{\sqrt{y}} = constant$$

which, by Fermat's Principle, is the path that *minimises travel time* in this medium. And by Section 3, the curve satisfying $\frac{sin\theta}{\sqrt{y}} = constant$ is precisely the cycloid.

So the bead and the light ray are governed by the same equation. The bead's speed matches the light speed in Bernoulli's imaginary medium; the time-minimising path for the light is the cycloid; therefore, the time-minimising path for the bead is also the cycloid. That is the brachistochrone.

To find the *right* cycloid for a given pair of points A and B, you simply choose the radius *r* so that the cycloid passes through B. Since $\frac{sin\theta}{\sqrt{y}} = \frac{1}{\sqrt{2r}}$, the constant in Snell's Law fixes *r* and *r* fixes the cycloid.



For the points A and B shown at the start of the essay, the path of shortest time – the brachistochrone – is shown above.

5. Conclusion

And there we have it; we have discovered the curve that minimises travel time from A to B! To summarise, we began with Fermat's Principle and Snell's Law, turned to the geometry of the cycloid, and finally brought these together to solve the brachistochrone problem.

What is perhaps most striking is how this result emerges so elegantly by considering a ray of light, as opposed to a physical object like a bead: an idea I personally found rather unexpected.

Thank you for your time and I hope you enjoyed this whistlestop tour!

Reference List

Desmos. (2024). Cycloid. [online] Available at: https://www.desmos.com/calculator/kaoeswcouq.

Levi, M. (2015). Quick! Find a Solution to the Brachistochrone Problem | SIAM. [online] Society for Industrial and Applied Mathematics. Available at: https://www.siam.org/publications/siam-news/articles/quick-find-a-solution-to-the-brachistochrone-problem/.

O'Connor, J.J. and Robertson, E.F. (2002). Brachistochrone problem. [online] Maths History. Available at: https://mathshistory.st-andrews.ac.uk/HistTopics/Brachistochrone/.

Sanderson, G. (2016). The Brachistochrone, with Steven Strogatz. YouTube. Available at: https://www.youtube.com/watch?v=Cld0p3a43fU.

Sanderson, G. (2023). Answering refractive index questions from viewers | Optics puzzles 4. [online] www.youtube.com. Available at: https://www.youtube.com/watch?v=Cz4Q4QOuoo8.