<u>This Will Blow Your Mind!</u> By George Costiff

We all remember in primary school numeracy lessons learning how to add numbers up; this all felt very simple and easy to understand. 1+1, 7+2, 3+3, all these are about as easy as maths gets. But then appears a *seemingly* innocent question... What is 1, add the next number, add the next number and so on and we keep doing this until the end of everything.

I can guarantee that the answer to this question will leave you as surprised as when you found out that 100 isn't the biggest number!

1.

But first, something that will leave your brain very much intact, I want to set the foundations of how we actually discovered addition. (This will all make sense later on). Imagine you have a banana, (apologies if you don't like bananas) and then another banana turns up. You can quite clearly see that there are 2 bananas. So, if we forget about the "*bananananess"* of the bananas, we get the idea of putting 1 and 1 together to get 2, which is addition. This is how maths can be discovered; we abstract numerical ideas from real life situations.

Before you think this is a world record attempt for the most times banana is said in a paragraph, it's time to blow your mind...

2.

I'll tell you that 1+2+3+4+.... Until infinity is in fact -1/12.

And I can prove it.

First of all, we need to know the value of Grandi's series (or sum):

$$1-1+1-1+1-1...$$

Quickly we can see that it will be equal to 1 or 0, depending on where we put the brackets:

$$(1-1)+(1-1)+(1-1)... = 0$$

And just as validly:

$$1+(-1+1)+(-1+1)+(-1+1)= 1$$

But which one is it? Well, we have to be a bit more clever about it.

Let this series be equal to S1. Next we will add S1 to S1 like this in a column method:

$$\begin{array}{cccccccc} 1 & -1 & +1 & -1 & +1 & -1 & & \\ & 1 & -1 & +1 & -1 & +1 & & + \\ \hline 1 & +0 & +0 & +0 & +0 & +0 & & \end{array}$$

Therefore:

$$S1 + S1 = 1$$

$$2(S1) = 1$$

$$S1 = \frac{1}{2}$$

Although this isn't what we expected, this does feel quite nice, as it is the mean of the two expected results.

The other series we have to understand is: 1-2+3-4+5-6… We do basically the same thing as before:

Let this series be S2.

1	-2	+3	-4	+5	-6…	
	1	-2	+3	-4	+5…	+
1	-1	+1	-1	+1	-1	

And we get grandi's series again, which equals a half.

So:

$$S2+S2 = S1 = \frac{1}{2}$$

$$2\,(S2) = \frac{1}{2}$$

$$S2 = \frac{1}{4}$$

Now we have everything we need for the proof that the sum of all the counting numbers (or natural numbers) is -1/12.

$$\text{Let } S = 1+2+3+4\ldots$$

We do S – S2:

1	+2	+3	+4	+5	+6	
1	-2	+3	-4	+5	-6	-
0	+4	+0	+8	+0	+12	

$$S-S2 = 4+8+12+\ldots$$

$$S-1/4 = 4\,(1+2+3+\ldots)$$

$$S-1/4 = 4\,(S)$$

$$-1/4 = 3\,(S)$$

$$S = -1/12$$

3.

Initially, I didn't want to accept that the sum of all the natural numbers was -1/12, despite the algebra being perfectly good, because this is so counter intuitive, that adding up lot's of positive, very big things, equals a negative, very small thing.

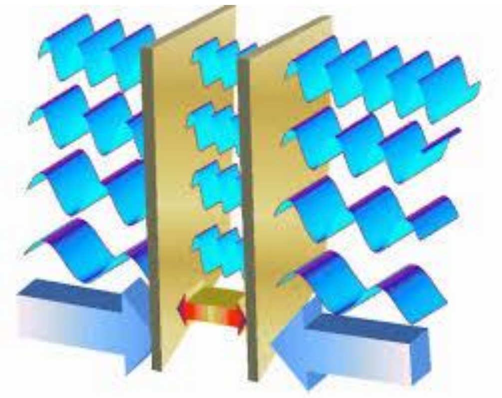
Now, using physics, I will attempt to reconstruct your brain and convince you that this is the case. This brings us back to the bananas. We can get to grips with the idea of addition, because we can see how it would work in a real life context. However, we cannot imagine a context where this series is equal to -1/12. But in Physics, the Casimir effect is exactly this:

It turns out, empty space isn't actually empty. (I promise this will be the end of me going against everything you thought about the universe). There are lots of tiny electromagnetic fluctuations with an infinitely different number of wavelengths that exist all around us, even in a vacuum. Therefore, if you put two, uncharged, metal plates, nanometers apart from each other, in a vacuum, there will be a little force between them called the Casimir force.

Here's a little analogy to help explain this. Imagine two boats sailing in the ocean very close to each other. Waves hit the outside of the boats, but the outside of the boats protect each other's inside from the waves around them. As a result, the force that moves them closer together (from the waves hitting the outside) is greater than the force that moves them apart- pushing them together. This is how the Casimir effect works.

In 1948, Casimir, a Dutch Physicist, discovered that only certain wavelengths can "fit" in between the two plates. However, any wavelength of fluctuation can exist outside the plates. This creates a pressure difference between the inside and outside of the plates. So there is a force, called the Casimir force, that moves the plates together. We call this attracting the plates together.

In Physics, we define attractive forces as negative, because they act to decrease the distance between the plates. Because the Casimir force is attractive, this is a real life demonstration of how infinities can sum to be something negative; there is an infinite number of different wavelengths, but they create a negative force.



4.

This is all very convincing that the sum of all the natural numbers is -1/12. However, some may disagree with this, because they have knowledge of standard results for summations. In particular, the result for:

$$\sum_{n=1}^{n} = \tfrac{1}{2} n (n+1)$$

For -1/12 advocates, this is quite worrying, as there is no sign of -1/12 in this what so ever. However, this all depends on how you regulate the sum. Regulating is a method used to calculate infinite sums. We assign finite values to help control the sum.

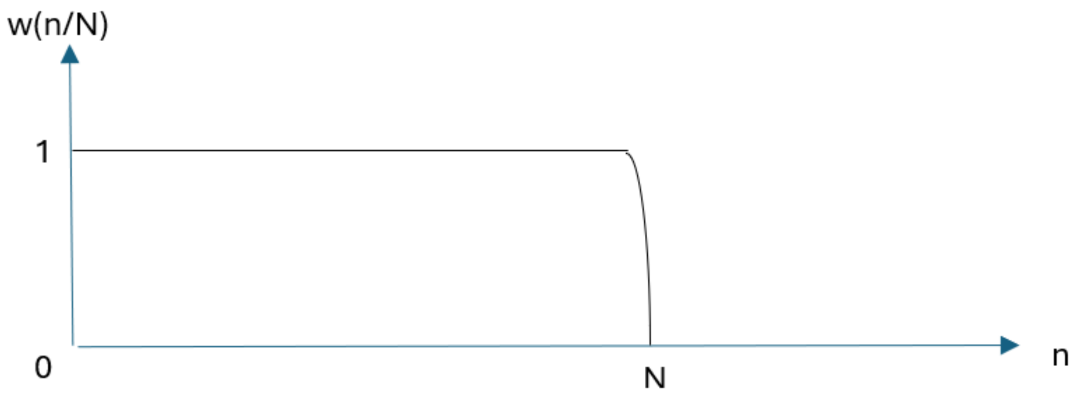
For example, we can cut this infinite series off at N: 1+2+3+4+…+N. What we have to do is multiply each term in the series by a weighting function: w(n/N), that becomes 0, when we are multiplying numbers we no longer want (as they have been "cut off"). So the sum we are doing is:

W(1/N)x**1** + w(2/N)x**2** + w(3/N)x**3** +…+ w(N/N)x**N** +w(N+1/N)x**(N+1)**…

The value of these weighting functions comes out to be 1, because we want to add up 1 lot of each of these terms.

This function is 0, because we want to cut it off at N, so we don't want to add anything else after it, and 0 lot's of anything is 0.

Because we regulate the sum using this weighting function, we are cutting it off very sharply and abruptly, because the function moves from 1 through to zero very quicky:

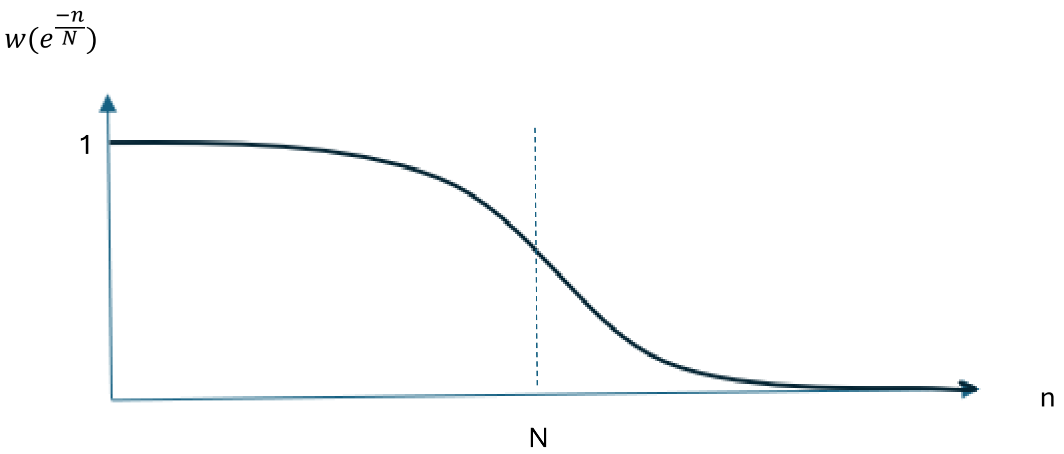


As a result of cutting the series off so abruptly, we don't get any sign of -1/12.

However, as Terence Tao, who is regarded as one of the greatest mathematicians, explains, if we regulate the sum using a weighting function that moves much more smoothly from 1 to 0, -1/12 does show up.

For example: $\sum_{n=1}^{\infty} n e^{\frac{-n}{N}}$ Goes from 1 to 0 nice and smoothly like this:

This is the weighting function



And we get: N^2- 1/12+ [a few very tiny numbers that are vanishingly small for infinitely big N].

And it turns out that this is very general, as different functions will all give numbers in the form: kN^2-1/12+[some small numbers], where k is a number that depends on the function.

In fact, there are some functions, that as N tends towards infinity, it throws out exactly -1/12, with no infinity or adjustments to it. For example: $\sum_{n=1}^{\infty} ne^{\frac{-n}{N}}\cos\left(\frac{n}{N}\right)$ = -1/12.

Now the question arises, that if different regulators give out different answers, which ones are the best?

To answer this, we need to look into particle Physics. Yep, it's come back again! Without going into too much of the nitty gritty of it, symmetry is a very important idea in it. We can think of symmetry as reflections where you look the same in the mirror, or rotational symmetry where you rotate a sphere and it is exactly the same as before the transformation. Basically, symmetry controls the structure of fundamental particles (such as quarks). So the main idea is that we really do not want to break it. However some ways of doing these infinite sums do break it. So, you lose control of the sum. So we ask ourselves, in which ways do we not break the symmetry?

It is when we have functions like $\sum_{n=1}^{\infty} ne^{\frac{-n}{N}}\cos\left(\frac{n}{N}\right)$.

Therefore it is the ones where we get exactly -1/12 with no corrections or spiraling off to infinity that give us the best answers.

(Quite a nice explanation for this is that it is nature's way of keeping us safe from infinity).

5. Conclusion

I hope you found this essay mind blowing, as after researching the proof that the sum of the natural numbers is -1/12, my mind was pretty blown, which was the inspiration for the title. And within this essay, I wanted to show you my thought process which followed, to help myself get to grips with this unbelievable concept, and what I found from my research.

As a fan of pure mathematics, I was glad to be able to use physics to convince everyone that this isn't just an interesting, ridiculous mathematical idea, that Pure Mathematicians have made up, but to show that this is very applicable in Science.

<u>References:</u>

Is Maths Real by Eugenia Cheng

Numberphile