Why a Piano Sounds Like a Piano

Or

Finding the *perfect* in the *imperfect*

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Movement 1: Introduction and The Harmonic Series

The piano is perhaps the most 'fundamental' of all instruments; and as a piano player and fan myself, I don't mean that in a bad way: there is something about this massive black box, with its 88 black and white keys, that became synonymous with music itself. Therefore, I find it most fascinating that, in this seemingly perfect feat of human construction and creativity, not a single note is truly in tune.

To answer why that is; how the perfect construction requires imperfection to work, we first must ask ourselves, 'why does a piano sound like a piano'.

At first, it might seem a silly question to ask; things sound the way they do and that's the way it is. But digging into the physics, the maths behind the piano can reveal what it sounds the way it does.

Before even looking at the piano we first have to understand the harmonic series. The harmonic series is the sequence of harmonics whose frequency is an integer multiple of the fundamental frequency (the lowest frequency produced, and the one used to name the note). In other words, when you play a note on any instrument, you hear not just the frequency of note you play, but also all its *overtones*. Where an overtone with frequency f_n as defined as:

$$f_n = nf_1 \tag{1.1}$$

Where f_1 is the fundamental. For example, the harmonic series for the note C2 can be shown below, where each note above the first is an *overtone* of the first note:

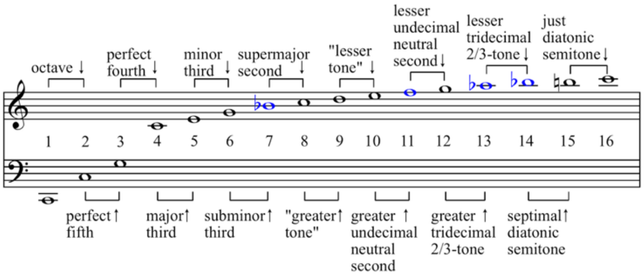

Figure 1

The harmonic series is exactly what gives other instruments their unique sounds, with the amplitudes of different harmonics giving different sounds to different instruments. For example, this is the difference between the amplitudes of the overtones in different wind instruments:

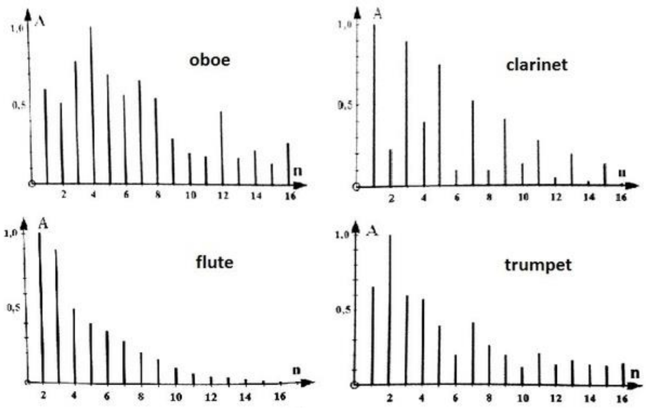

Figure 2

However, for a piano, this is not the main reason why it has a distinct tone so different to all other instruments. To find the reason for that, we must look deeper into the mathematics and physics behind the piano.

Movement 2: Inside the Piano

First, let us investigate the inside of our grand piano. A quick look will show us the different strings inside the piano, with strings increasing in length when moving towards the left (lower notes) of the keyboard. One's first question may be 'why is that?'.

In physics, the definition of a standing wave is a wave which oscillates in time but does not propagate through space. The piano string provides an example of a standing wave – a fixed string at both ends which, when struck, vibrates. The velocity of string vibration is governed by the central formula:

$$v = \sqrt{\frac{T}{\mu}} \qquad (2.1)$$

Where v is the velocity of propagation of a wave, T is the force of tension in the string and μ is defined as the linear density of the string or:

$$\mu = \frac{m}{L} \tag{2.2}$$

Where m and L are the mass and length of the string respectively. Therefore, since the velocity of propagation is also equal to a wave's wavelength (λ) multiplied by its frequency (f), and since the fundamental harmonic is defined as double the length of the string, we can derive Mersenne's law:

$$v = \lambda f \tag{2.3}$$

$$\lambda = 2L \tag{2.4}$$

$$f = \frac{1}{2L}\sqrt{\frac{T}{\mu}} \tag{2.5}$$

In the standard tuning system (12 tone equal temperament), one note is double the frequency of the note an octave below it e.g. the frequency of A4 is 440Hz while the frequency of A3 is 220Hz. Using the equation above, we can see that, to get a note to be an octave lower, we need to double the length of the string. This produces an expected length vs note graph like so:

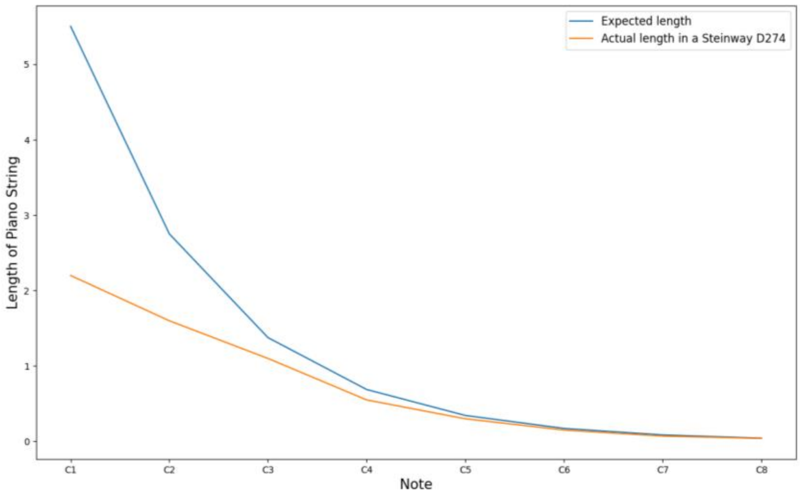

Figure 3

This shows how a piano would need to be more than 5m long in order to produce the sound of its lowest notes! How did we get around that?

The answer lies in increasing the linear density of the string (as using (5) will show that the higher μ is, the lower the frequency of the string) which is why you may notice that the thickness of the piano string increases as the pitch decreases, with the lower few notes being covered with wound copper wire.

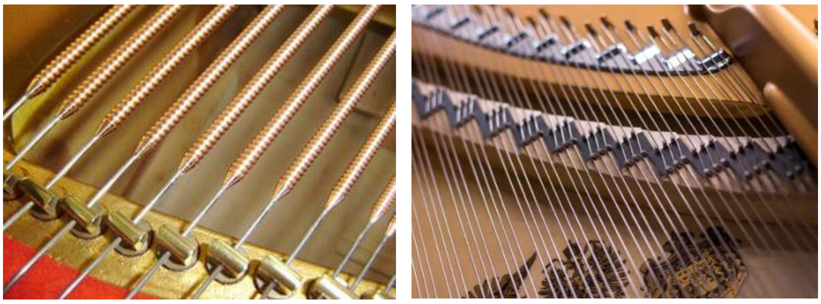

Figure 4 – the left shows the thicker lower notes while the right shows the thinner upper ones

The thickness of the strings, particularly of the lower ones, is a major reason for the piano having its characteristic sound, as they contribute to the string stiffness. To understand this, we must look at the mathematical definition of a standing wave as well as the wave equation.

To begin, we will define the equation which describes the motion of waves – the wave equation:

Intermezzo (Movement 3): The Wave Equation (DUN DUN DUUUUN!)

$$\begin{cases} \dfrac{\partial^2 y}{\partial t^2} = v^2 \dfrac{\partial^2 y}{\partial x^2} & x \in (0, L), t > 0 \\ y(0,t) = y(L,t) = 0 & t \geq 0 \\ y(x,0) = \phi(x) & x \in [0, L] \\ \partial_t y(x,0) = \psi(x) & x \in [0, L], \end{cases} \tag{3.1}$$

Where $y(x,t)$ represents the displacement of string of length L from its equilibrium position, $\phi(x)$ and $\psi(x)$ represent the initial shape and velocity of the string respectively. This model is a great approximation for the strings of say, a guitar or cello, as their strings are almost perfectly flexible. However, piano strings, especially towards the lower keyboard, have much greater stiffness, and the classical wave function needs to be adjusted. To do this, we must first adjust the equation to represent forces and not velocity.

Using equation (2.1) we can rearrange the wave equation into this form:

$$\mu \frac{\partial^2 y}{\partial t^2} = T \frac{\partial^2 y}{\partial x^2} \tag{3.2}$$

Thus, this becomes an extension of Newtons second law $F = ma$. With the left side representing ma and the right side representing the restoring force due to tension. To consider the total restoring force due to the string's stiffness, we must first look to the Euler-Bernoulli beam theory:

$$\frac{\partial^2}{\partial x^2}(EI \frac{\partial^2 y}{\partial x^2}) = q(x) \tag{3.3}$$

Where q(x) the force per unit length, E is the young's modulus of the string and I is the second moment of area of the string defined by:

$$I_x = \int y^2 \ dA$$

Where (3.4)

$$y = r \sin \theta$$

$$dA = r \ dr \ d\theta$$

Representing the cross section of the string (a circle) in polar coordinates. Solving and plugging into (3.3) we get:

$$I = \frac{\pi R^4}{4} \tag{3.5}$$

$$\frac{\partial^2}{\partial x^2}(E\frac{\pi R^4}{4}\frac{\partial^2 y}{\partial x^2}) = q(x) \tag{3.6}$$

Where R is radius of the string. Considering that the cross-sectional area of the string is:

$$S = \pi R^2 \tag{3.7}$$

And the radius of gyration (K), for a cylindrical shape of radius R, is equal to:

$$K = \frac{R}{2} \tag{3.8}$$

We can present the entire equation for restoring force due to stiffness as:

$$-ESK^2 \frac{\partial^4 y}{\partial x^4} = q(x) \tag{3.9}$$

The negative sign is there since the equation for the Euler-Bernoulli theory represents the internal bending moment resisting deformation, so the negative sign ensures that it is an additional restoring force that will act alongside our tension, rather than opposite it.

Therefore, our final wave equation, with the consideration of the stiffness of the string, is:

$$\mu\frac{\partial^2 y}{\partial t^2} = T\frac{\partial^2 y}{\partial x^2} - ESK^2 \frac{\partial^4 y}{\partial x^4} \tag{3.10}$$

The reason for getting this equation will become clear in a minute, but firstly we need to solve our equation, using the general solution for a standing wave:

$$y(x,t) = \sin\frac{n\pi x}{L}e^{i\omega t} \tag{3.11}$$

Where n is nth harmonic ω is the angular frequency. Now we can solve this equation by, firstly, assuming a normal mode i.e. one where every point moves with a fixed phase relation to the other:

$$y(x,t) = X(x)T(t) \tag{3.12}$$

Substituting into the partial differential equation:

$$XT'' = v^2 X''T - \frac{ESK^2}{\mu} X''''T \tag{3.13}$$

Dividing both sides by XT and following through by finding the derivatives to both sides will give:

$$\omega^2 = v^2 \left(\frac{n\pi}{L}\right)^2 + \frac{ESK^2}{\mu} \left(\frac{n\pi}{L}\right)^4 \tag{3.14}$$

We can factor and take square roots to find:

$$\omega = \frac{n\pi v}{L} \sqrt{1 + \frac{ESK^2}{\mu v^2} \left(\frac{n\pi}{L}\right)^2} \tag{3.15}$$

And since

$$f_n = \frac{\omega_n}{2\pi} \tag{3.16}$$

We rearrange to get:

$$f_n = \frac{nv}{2L} \sqrt{1 + \frac{ESK^2}{\mu v^2} \left(\frac{n\pi}{L}\right)^2} \tag{3.17}$$

Lastly, we can use $f_1 = \frac{v}{2L}$ to get:

$$\boldsymbol{f_n = nf_1\sqrt{1 + Bn^2}} \tag{3.18}$$

Where $B = \frac{ESK^2}{\mu v^2} \left(\frac{\pi}{L}\right)^2 = \frac{ESK^2\pi^2}{TL^2}$.

Movement 4 and Finale: Tying it all together

Now we have it! After considering the thickness of the string, we found that, in our stiffer piano strings, the harmonic series gets distorted and does not behave in the clean $f_n = nf_1$ as it might on a guitar or clarinet. In music, these distorted overtones are called partials (although the terms overtones and partials are interchangeable in this case), and we can show visually:

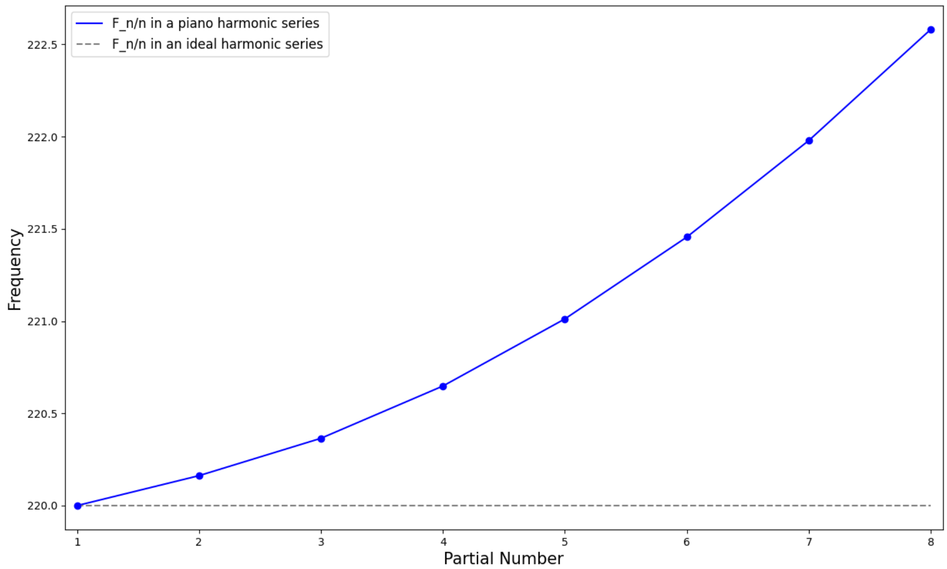

Figure 5

This graph shows the value of a frequency of a partial n divided by n in the A3 string, which in a regular overtone series would always give the frequency of the fundamental (as defined by $f_n = nf_1$). However, we can see that, in actuality, the overtones have a higher frequency and are distorted, particularly as the partial number increases.

This is why piano tuners often exhibit what is called 'octave stretching' as shown in the figure below (called the Railsback curve):

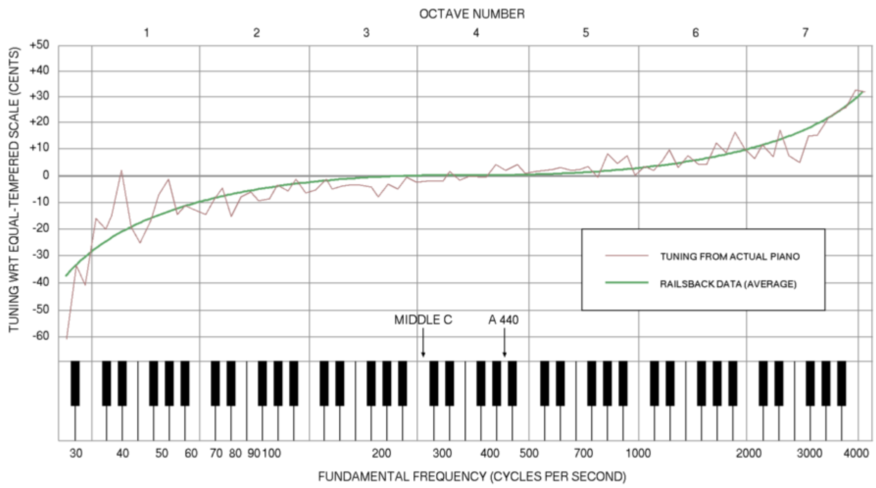

Figure 6

What is shown here is the difference in cents (a unit where the difference in frequency between two keys on a piano = 100 cents) between the fundamental frequency of a note in theory and its fundamental frequency on an average piano, with the lowest note on the piano – the A0 – being almost more than a third of a note lower than it theoretically should be.

The graph shows that the upper notes (above C4) get stretched upwards while the lower notes (below C4) get stretched downwards. This is because, the upper notes are tuned by 'partial matching', where each note is tuned to match the second partial of the note below it. This makes sense as, referring to our harmonic series, we can see that the second partial of a note is identical to the note an octave above it (i.e. the second harmonic of a C4 is a C5). However, the stiffness of piano strings means the second harmonic of any given note is a few cents higher than it 'should be' and so, the C5 on the piano will be 'stretched upwards' as it matches the second partial of the C4. This difference stacks and, by the time you tune C8, you are almost a third of a note higher than 'true C8', explaining the increasing upwards stretch in the upper notes of the piano.

In contrast, the lower notes are stretched down as, referring to our equations (3.18), we can see that:

$$f_n = nf_1\sqrt{1 + Bn^2}$$

Where:
$$B = \frac{ESK^2}{\mu v^2}\left(\frac{\pi}{L}\right)^2 = \frac{ESK^2\pi^2}{TL^2}.$$

Since the lower notes are stiffer (due to their larger linear density), their young's modulus (E) is greater and therefore their overtone series is more distorted. To 'fix' this, notes are stretched down so their partials more closely match the 'true' overtones of the note, at the cost of changing the fundamental of that note. As a whole phenomenon, we call the stretching of the piano notes 'inharmonicity' and *that* is precisely why our piano sounds like a piano, as it provides the signature warm, wooden tone.

Therefore, we have finally reached an answer to our question, the reason for the piano's warm, distinct tone is precisely due to its 'inharmonicity', where the fundamental notes themselves are not tuned to their 'true' sound. Perhaps it is important for us to remember, in our everyday lives, that even the symmetrical, ordered piano with its 88 keys and slick design is itself reliant on the imperfect, the chaotic, the slight error, to make beautiful music.

Thank you for reading, hope you enjoyed!

Appendix 1: Python Codes

Figure 3:

```
import matplotlib.pyplot as plt
import numpy as np

l = 0.043
c_string_lengths_m = [2.2, 1.6, 1.1, 0.55, 0.30, 0.15, 0.07, 0.04]
values = []
notes = []
values.append(l)
notes.append('C8')
for i in range(8,1, -1):
    note = "C" + str(i-1)
    l = l*2
    values.append(l)
    notes.append(note)
values.reverse()
notes.reverse()
plt.plot(notes, values)
plt.plot(notes, c_string_lengths_m)

plt.xlabel('Note', fontsize = 15)
plt.ylabel('Length of Piano String', fontsize = 15)
plt.legend(['Expected length ', 'Actual length in a Steinway D274'], fontsize = 12)
plt.show()
```

Figure 5:

```
import matplotlib.pyplot as plt
import numpy as np

E = 2.0e11
f1 = 220
L = 1.3
r = (0.9e-3)/2
rho = 490
mu = rho * np.pi * r**2
T = (2 * L * f1)**2 * mu
I = (np.pi * r**4) / 4
B = (np.pi**2 * E * I) / (T * L**2)
f1 = 220
frequencies = [f1]
base = []
partials = [1,2,3,4,5,6,7,8]

for n in range(2,9):
    fn = n*f1*np.sqrt(1+B*n**2)
    fn = fn/n
    print(fn)
```

```
    frequencies.append(fn)
for i in range(1,9):
    base.append(220)

plt.plot(partials, frequencies, color = 'blue')
plt.plot(partials,base, '--', color = 'grey')
plt.plot(partials,frequencies, 'o', color = 'blue')
plt.xlim(0.9,8.1)
plt.xlabel('Partial Number', fontsize = 15)
plt.ylabel('Frequency', fontsize = 15)
plt.legend(['F_n/n in a piano harmonic series', 'F_n/n in an ideal harmonic series' ], fontsize = 12)
plt.show()
```

Appendix 2: Bibliography

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