Is i truly imaginary?

In all of mathematics we were told that $\sqrt{(-1)}$ couldn't be done, we were told that there were no possible solutions and no way to solve this. Later on however we learnt that it actually can be done if we just labeled it as an imaginary constant 'i', but why do we label this as imaginary when it is just as real as all the other numbers and constants in maths. In maths it is said that no two numbers that are the same can multiply together to give a negative result, with this constant i, it was all possible. For centuries many mathematicians deemed this to be impossible and rejected the idea entirely.

History behind i:

The first occurrence of i was found by a mathematician that goes by the name of Gerolamo Cardano. He proposed the problem 'find two numbers that add to 10 and multiply to give 40' which lead to the equation:
$x^2 - 10x + 40$ being made. Using his formula he obtained two solutions. $x = 5 + \sqrt{(-15)}$ $x = 5 - \sqrt{(-15)}$. He verified that these technically solved the solution but he called the result 'sophistic' and deemed them to be useless although he used it, he still refused to accept it.

Then came the first true believer, Rafael Bombelli. He was an Italian engineer who published L'Algebra in 1572, and he did something that was unspoken of at the time, he decided to take $\sqrt{(-1)}$ seriously and used it in all his calculations. He was attempting to solve the cubic $x^3 = 15x + 4$. Cardano's formula gave him the results of:
$x = \sqrt[3]{(2 + \sqrt{(-121)})} + \sqrt[3]{(2 - \sqrt{(-121)})}$. This was very disturbing for Bombelli as the equation clearly has a solution $x=4$, so what confused him was how this mess of square and cube roots of negative numbers equals a

simple whole number. However instead of dismissing them as being fake solutions he took a big leap of faith and guessed that the solutions might be in the form of $a + b\sqrt{(-1)}$. When he added the two parts of the previous solution he found the imaginary parts cancelled out perfectly. He concluded that $\sqrt{(-1)}$ wasn't nonsense, but instead it was a consistent mathematical object that actually obeyed the rules of maths.

Further down the line there were many mathematicians that used i in their work. Abraham de Moivre was the mathematician who made the revolutionary formula $(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)$. This shows that it wasn't just a fun theory, but an important constant that links trigonometry and geometry.

Finally there was Leonhard Euler, the man who gave $\sqrt{(-1)}$ a name (i). He also gifted the world with the formula: $e^{i\theta} = \cos\theta + i\sin\theta$/ Euler used i with complete confidence and no philosophical hesitation.

What is i?

As mentioned earlier i is just the square root of -1, but what does that really mean? Well you can basically consider it as merely a step forward in terms of number systems. Natural numbers, being counting numbers (1,2,3,4), whole numbers that include 0, integers being any whole number and their negative counterparts. Rational numbers being numbers that can be expressed as fractions like p/q, this includes recurring decimals. Irrational numbers being numbers that cannot be expressed as a fraction, the primary examples are pi and root 2. And the final classification being real numbers which takes into account any number rational, irrational and on a number line. Each step onto the next number classification brings along another level of complexity. And with the addition of the complex number classifications, being any number in the form of a +bi we break the very law that defines our number system. With this the complex classification is the end of the line as there is no further extension needed for polynomials. Creating this variable i aided in accomplishing the final step of solving polynomials.

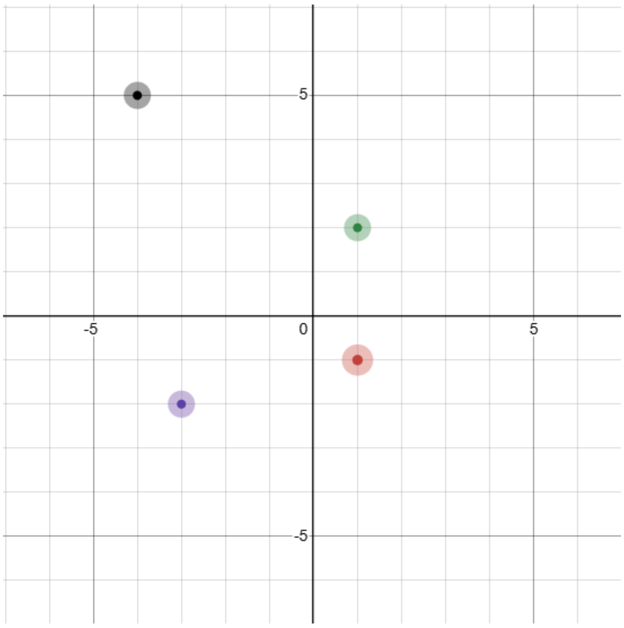
The algebra behind complex numbers

Just like other number classifications it has its own rules when it comes to its properties and algebra.
Considering $i^2 = -1$ the complex classification has some interesting properties. As known, a complex number is defined as '$a + bi$' where a and b are constants and i is $\sqrt{-1}$. When you add two complex numbers, for example: $(a + bi) + (c + di) = (a + c) + (b + d)i$. This basically treats i as an algebraic expression rather than something bizarre. For multiplication it gets slightly interesting, if we consider: $(a + bi)(c + di)$ we get $(ac - bd) + (ad + bc)i$. The reason this is interesting is because when we multiply out the expression we get bdi^2, and as known $i^2 = -1$, so this expression becomes $-bd$, which makes it into a real number. Additionally, in the complex plane, each complex number has a complex conjugate, this is where the sign of the complex part of the number is changed. For example, consider $a + bi$, its complex conjugate would be $a - bi$. This is interesting because in a polynomial, complex solutions always come with its conjugate pair. The reason for this is because the coefficients of the polynomial must remain real, so having a complex number multiplied with its conjugate cancels out the imaginary part. It can be thought of as the difference between two squares. There is also the modulus, which is basically the 'length' of a complex number, this is found by multiplying the complex number by its complex conjugate, it can also be found by using pythagorean theorem, if you just consider the real and imaginary coefficients as the opposite and adjacent of a right angled triangle, giving you the 'length' of the hypotenuse. This 'length' is a part of the complex plane which I

will talk about momentarily. Another interesting property of i is that there is a cycling nature behind it. For example, consider i, i^2, i^3 and i^4. Each time you are increasing the power by 1. If you equate these complex numbers something interesting occurs. i will stay as i, i^2 will go to -1. i^3 will go to -i and i^4 will go to 1. This creates a cycling period of 4 which hints on a rotational nature of i.

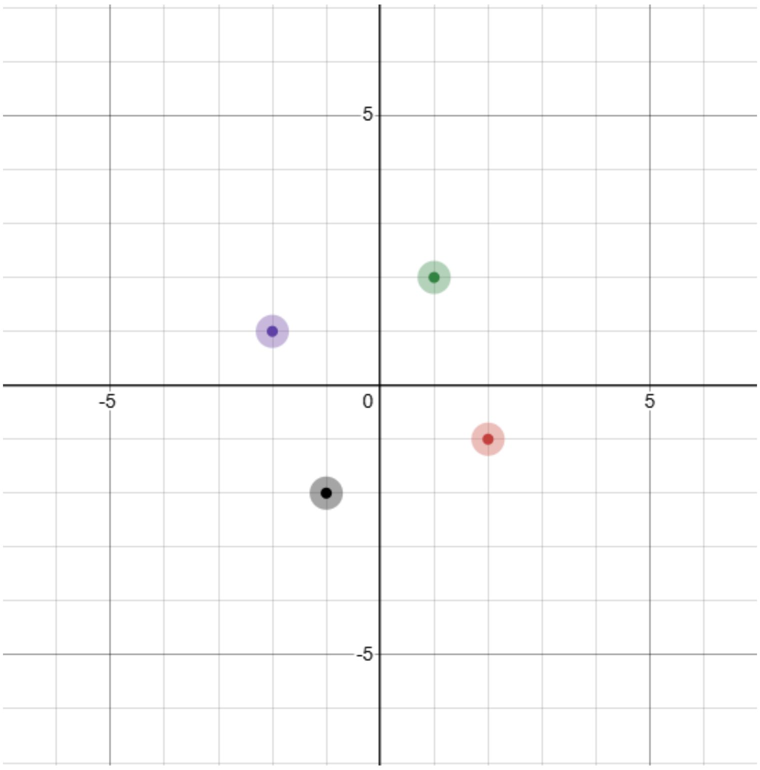
The geometry behind i

Going deeper into the world of complex numbers, there is actually a way to visualise what complex numbers would look like by plotting them on an argand diagram. An argand diagram is basically representing a complex number (a + bi) as x and y coordinates, with the x axis being the real part of the complex number and the y axis being the imaginary part of the complex number. In this context, a would be the x value and b would be the y value. As shown on the right, there are four points plotted onto the diagram according to their real and imaginary coefficients.



Green point: 1 + 2i
Purple point: -3 -2i
Black point: -4 + 5i
Red point: 1 - i

As mentioned earlier you can get the modulus of a complex number by rooting the sum of the squares of the coefficients of a complex number, giving a value of its ‘length’, with an argand diagram we can visualise this ‘length’. Consider the green point on the argand diagram, (1 + 2i), its modulus would be √(1^2 + 2^2) which equates to √5 which is approximately 2.236. On an argand diagram, this value represents the distance between the origin and the green point, being 2.236 units in length. The interesting thing about this is that we went from √(−1) to a distance that almost feels real. Another thing that was previously mentioned is that there is a rotational nature to i, the interesting thing about this is that it can be represented on an argand diagram. Consider the complex number (1 + 2i).



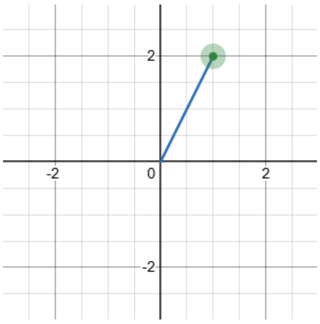
Green point: 1 +2i

Purple point: -2 +i

Black point: -1 - 2i

Red point: 2 - i

Each time we multiply the new point by i it rotates 90 degrees until it is back in the original position (the green point). Another aspect we can obtain from a complex is its argument, this is the angle in radians between the real x axis and the line segment formed from the modulus of a complex number. Consider the green point and its line segment drawn on the argand diagram on the right. We can model this as a triangle in order to get this argument. If we take θ to be the angle



between the real x axis and the line segment (θ being the argument in this case) we can take the opposite length to be 2 units and the adjacent length to be 1 unit. With this we can use the formula tan(θ) = 2/1 if we use the inverse function of tan we get a value of 1.107 radians. With the modulus and the argument of a complex number we can put it into polar form being r(cosθ +isinθ) where r is the modulus and θ is the angle measured in radians. As you can see just from taking a visual representation of √(−1) we have nicely linked it with sine and cosine functions which are very relevant in our society today for many purposes.

Euler's formula, the crown jewel:

Previously it was shown how a complex number can be written in terms of sine and cosine, with this we can take it a step further by introducing Euler's number into the mix of things. But what does e have anything to do with √(−1)? Surprisingly it all makes plenty of sense. Considering that e is the constant that relates to growth, when we input i into it, it results in a bridge between exponential growth and trigonometry. If we take the Taylor series functions sin(x), cos(x) and e^x. Taylor series being a function being represented as an infinite sum of polynomials.

- **Exponential:** $e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \dots$
- **Cosine:** $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$
- **Sine:** $\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \dots$

Now when we replace the x in e^x with iθ we get:

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \frac{(ix)^5}{5!} + \dots$$

From this new function we can separate it into two parts, real and imaginary parts as shown: $e^{ix} = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right)$

As you can see now, the real part of the series is the same as the series for cos(x) and the imaginary part of the series is the same as the series for sin(x). This gives us the polar formula re^iθ = r(cosθ + isinθ) as proved from the Taylor series. Now when we have θ as π, we get a special case where e^iπ = -1 which creates the famous Euler formula e^iπ +1 = 0 which connects the five most fundamental constants in mathematics into one equation, all of which are from completely different fields of maths.

From this we have given something that is not even possible and even considered imaginary a real purpose in mathematics, making it the glue of maths.

Is i imaginary?

Back to the original question in all this, does i deserve to be degraded as being imaginary. The answer is that it's just as real as all the other numbers, it is but a different type of number. From i we have been able to bridge the world of geometry and rotation to exponential growth, because of i we are able to simplify complex trigonometric functions, making them easier to understand in calculus. But this begs the question, are there more imaginary numbers out there? Another well known impossible calculation in mathematics is 1/0. The logical answer to this would be infinity as if we graph this as a function the y value tends to towards infinity. But if we use common sense we know that no amount of nothing can create something. But as we know, when √(−1) was first introduced everyone thought it was bizarre and couldn't be done and look where it brought us.