Touching on Untouchable Numbers

Poppy Ellis
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1 Introduction

In order to explore the concept of an untouchable number, we first must define one. A simple definition of an untouchable number is an integer that cannot be represented as the sum of the proper divisors of another integer (or itself, in the case of 28). It is believed that untouchable numbers were first discovered by Abu Mansur al-Baghdadi, a scholar from Baghdad, who taught arithmetic and had a particular interest in number theory, showcased in his book *al-Takmila fi'l-Hisab* (The Completion of Arithmetic). The proper divisors of a number refer to all of the integer factors of that number, excluding itself - the proper divisors of 8, for example, would be 1, 2 and 4. If a number is touchable, it can be represented by $n = \sigma(x) - x$, where $\sigma(x)$ is used to denote the sum of divisors function. For example, if we let x equal 8, the sum of the proper divisors of 8 - also referred to as 8's aliquot sum - would be 1 + 2 + 4, which is equal to 7. Therefore, as there exists such x that $7 = \sigma(x) - x$, 7 must be a touchable number.

However, if we take the number 5, we find that there is no such x that exists so that $5 = \sigma(x) - x$. We can prove this logically as well, for 5 to be touchable, there must be a number that exists whose factors (excluding itself) add to 5. Since repeat factors do not exist, the only sets of numbers that add to 5 are (1,4) or (2,3). Immediately we can rule out the (2,3) case as the proper divisors of any number would include 1. We can also rule out the (1,4) case as all numbers that have 4 as a divisor must also have 2 as a divisor, hence 1,4 cannot exist as a set of proper divisors. Therefore, there is no way to express 5 as the aliquot sum of another number, and it must be *untouchable*.

2 Properties of untouchable numbers

The first few elements in the set of untouchable numbers are 2, 5, 52, 88, 96, 120, 124… . At a glance, there seems to be no distinct pattern to them; they are unevenly distributed throughout the number set. Nevertheless, there are a few distinctive properties that nearly all of the untouchable numbers share. Firstly, with the exception of 5, all untouchable numbers are projected to be even numbers. This can be proven if we assume the Goldbach conjecture, which hypothesises that every even number greater than 2 can be written as the sum of two primes - but more importantly that every even number greater than 6 can be written as the sum of two distinct primes.

If we let an even number greater than 6 be equal to 2n, we can say that $2n = p + q$, where p and q are distinct prime numbers. Then, if we look at the number pq, the aliquot sum of pq will be equal to p + q + 1, which is 2n + 1. Therefore, every odd number greater than 7 must be touchable. We can quickly find ways to prove that the numbers 1, 3 and 7 are touchable - (the aliquot sum of 1 = 1, the aliquot sum of 2 = 1 + 2 = 3, and the aliquot sum of 8 = 1 + 2 + 4 = 7), thus 5 must be the only odd untouchable number. This proof also tells us that 2 and 5 are the only prime untouchable numbers, as every other prime number is an odd number greater than 7 hence must be touchable.

Another property of untouchable numbers is that no untouchable number is equal to p + 1, where p is a prime number. If we consider p^2, the proper divisors of that number will be p and 1, therefore the aliquot sum will be equal to p + 1, meaning all numbers that satisfy n = p + 1 will be touchable.

3 Positive Density of untouchable numbers

Another feature of untouchable numbers is that, unlike prime numbers, they have a positive number density, proven by Erdos in 1973. This means that throughout the integer set, a positive (non-zero) proportion of all numbers are untouchable. In 2011, Chen and Zhao proved that the density of untouchable numbers was at least 6%.

They did this by focusing on even untouchable numbers, partitioning the even numbers into sets, one containing all the even numbers up to a fixed integer M, and one containing all of the even numbers up to M which are touchable. They then prove that the first set is sufficiently big so that when the second set (of even touchable numbers) is subtracted, there is still a large proportion of even untouchable numbers.

They find a lower bound for the set of even numbers and an upper bound for the set of untouchable even numbers using Euler's totient - a count of how many numbers up until N are coprime to N - and then show, for an M with enough factors, the set of even numbers will always be far larger than the set of untouchable even numbers. They chose their M as $2^6 \times 3^5 \times 5^4 \times 7^3 \times 11^2 \times 13 \times 17 \times 19 \times 23 \times 29 \times 31 \times 37 \times 41$, so it has a sufficient numbers of divisors for their proof to hold, as the basis of the proof relies on the fact that

$4d < \sigma(2d)$

where d is a divisor of M. For a number with few divisors, take d = 1

$4d = 4$
$\sigma(2d) = \sigma(2) = 2 + 1 = 3$
Therefore 4d is greater than σ(2d) and the proof doesn't hold.

Take d = 6, a number with many more divisors

$4d = 24$
$\sigma(2d) = \sigma(12) = 1 + 2 + 3 + 4 + 6 + 12 = 28$
$4d < \sigma(2d)$, therefore the proof holds when a number has a large number of divisors.

At the end of their proof they are left with g_M, the density of the set of untouchable numbers, to be equal to 0.0602757, showing that 6% of all integers are untouchable numbers.

4 Other interesting types of numbers

It's also interesting to observe that certain sets of numbers, including perfect numbers and amicable numbers, can never be untouchable numbers - as their properties are based on being the sum of the proper divisors of some other number. An even more interesting case of numbers that can never be untouchable, however, are sociable numbers.

Sociable numbers form what is known as an aliquot sequence. This means that the sum of the proper divisors of some number A is equal to B, the aliquot sum of B is equal to C, the aliquot sum of C is equal to D and so on until you return to A. For example, take the number 12,496, its aliquot sum is 14,288, the aliquot sum of that is 15,472, its aliquot sum is 14,536, then 14,264, and then its aliquot sum goes back to 12,496. This is an aliquot pattern with order 5. This particular set was discovered in 1918 by Paul Poulet, and was also the first set of sociable numbers ever discovered. Poulet also discovered an aliquot pattern with order 28, this is the highest known magnitude of an order of aliquot pattern, and starts with the number 14,316.

Amicable numbers are a special (very friendly) subset of sociable numbers, with an aliquot pattern of order 2. Hence $\sigma(a) - a = b$ and $\sigma(b) - b = a$. However, normally sociable numbers refer to aliquot patterns with a higher magnitude than 2. The smallest amicable numbers known are 220 and 284: $\sigma(220) - 220 = 1 + 2 + 4 + 5 + 10 + 11 + 20 + 22 + 44 + 55 + 110 + 220 - 220 = 284$ and $\sigma(284) - 284 = 1 + 2 + 4 + 71 + 142 + 284 - 284 = 220$. Amicable numbers have an ancient background, mathematicians in Pythagorean Greece were aware of the first pair, 220 and 284.

Another type of number that has no crossover with untouchable numbers are Mersenne primes. Mersenne primes are prime numbers that fit the requirement $M_p = 2^p - 1$, where p is also a prime number. They are a rare type of number with only 52 known values, the smallest being 3 (2^2 -1) and largest known (as of right now) being $2^{136279841} - 1$. The proof that they must be touchable uses the fact that for every Mersenne prime there is also a number $n = 2^p$. If we consider $\sigma(2^p)$:

$\sigma(2^p) = 1 + 2 + 4 + \ldots + 2^{p-2} + 2^{p-1}$

This forms a geometric series with first term 1 and common ratio 2. Using $S_n = a(1-r^n) / 1 - r$, we can show that S_n is equal to $2^p - 1$, therefore all Mersenne primes are touchable numbers.

5 Conclusion

Unlike other sets of numbers, used in encryption and the natural world, untouchable numbers rarely exist outside of number theory meaning they are less heard of and overlooked. I thought there was a peculiar beauty in their definition and I found it very interesting to delve into this particular set of numbers. As well as this, it was also entertaining to take a quick glimpse at other, completely different, sets of numbers like sociable and amicable numbers. Furthermore, reading and taking in Chen and Zhao's proof (which would take a whole other essay to discuss!) was highly enriching, especially to pick up on the clever details they implemented (e.g. making their M a product of primes) in order to come to their final conclusion. Overall, untouchable numbers are a fascinating piece of maths that people do not know much about, therefore it was a great opportunity to learn a little more about something new.