How to Remove Everything and Still Have Infinity Left

A mathematical journey into the Cantor set

Sanny Yadav

An essay on infinity, length, and one of mathematics' most unreasonable objects.

Introduction: The Magician's Eraser

Imagine a magician standing before a blackboard. He draws a single, solid white line: the interval $[0, 1]$. Then, with a flourish, he erases the middle third. He erases the middle third of the two remaining pieces, and then the middle third of those four pieces, and then again, and again, forever.

A casual observer might conclude that, eventually, the line simply vanishes. After all, if we keep removing parts of something, there should eventually be almost nothing left. Continue long enough, and one expects complete disappearance.

And yet mathematics has a habit of taking the obvious, turning it inside out, and calmly insisting that the truth is stranger.

What remains after this endless erasing is called the *Cantor set*. It is not a line in the usual sense. It has no intervals, no visible bulk, and, in a precise sense, no length at all. And yet it is not empty. In fact, it contains infinitely many points — so many, in fact, that they cannot even be listed one by one.

This is not merely a curiosity. The Cantor set is one of those rare objects in mathematics that feels like a direct challenge to common sense. It is almost nothing, and yet far too large to ignore.

This essay is an exploration of that strange object: how it is built, why it has length zero, why it still contains uncountably many points, and what it reveals about the uneasy relationship between geometry and infinity.

Building the Cantor Set

The construction begins with the closed interval

$$[0,1].$$

Now remove its open middle third:

$$\left(\frac{1}{3},\frac{2}{3}\right).$$

What remains is

$$\left[0,\frac{1}{3}\right]\cup\left[\frac{2}{3},1\right].$$

Then we repeat the same operation on each remaining interval. From $\left[0,\frac{1}{3}\right]$, we remove its middle third $\left(\frac{1}{9},\frac{2}{9}\right)$, and from $\left[\frac{2}{3},1\right]$, we remove $\left(\frac{7}{9},\frac{8}{9}\right)$.

This leaves four intervals. Then we do the same thing again. And again. And again, forever.

The Cantor set is the collection of all points in $[0,1]$ that are *never* removed at any stage.

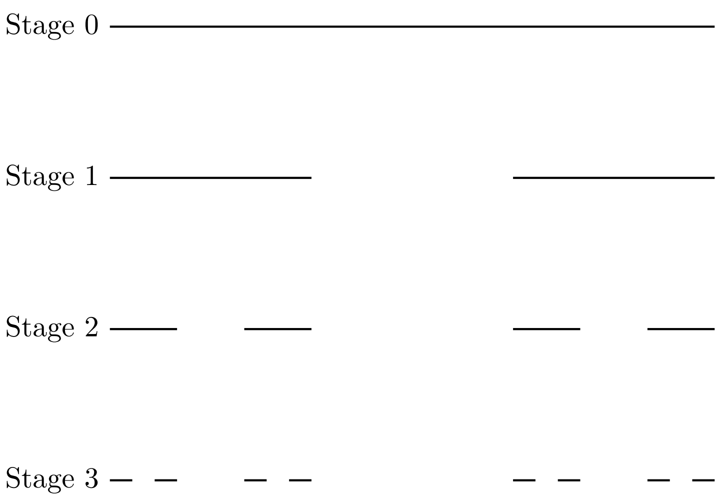

Figure 1: The first stages in the construction of the Cantor set.

At this point, a reasonable question presents itself: if we keep removing pieces forever, what could possibly survive?

That is exactly the right question.

First Surprise: It Is Not Empty

The Cantor set is certainly not empty.

For example, the endpoints 0 and 1 are never removed. Nor are $\frac{1}{3}$ and $\frac{2}{3}$. In fact, every endpoint created during the construction survives. So even after infinitely many cuts, there are still points left behind.

At first, this may not seem too dramatic. Perhaps only a few stubborn survivors remain — mathematical debris after a particularly aggressive demolition.

But the Cantor set is much richer than that. To see why, we first need to understand something even stranger: although points remain, the set has *no length at all*.

Second Surprise: It Has No Length

Let us calculate how much of the original interval is removed.

At the first step, we remove an interval of length

$$\frac{1}{3}.$$

At the second step, we remove two intervals, each of length $\frac{1}{9}$, so the total removed is

$$2 \cdot \frac{1}{9}.$$

At the third step, we remove four intervals, each of length $\frac{1}{27}$, so the total removed is

$$4 \cdot \frac{1}{27}.$$

Continuing in this way, the total length removed is

$$\frac{1}{3} + 2 \cdot \frac{1}{9} + 4 \cdot \frac{1}{27} + 8 \cdot \frac{1}{81} + \cdots .$$

This is a geometric series:

$$\sum_{n=0}^{\infty} \frac{2^n}{3^{n+1}} = \frac{1}{3} \sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n .$$

Since

$$\sum_{n=0}^{\infty} \left(\frac{2}{3}\right)^n = \frac{1}{1 - \frac{2}{3}} = 3,$$

we obtain

$$\frac{1}{3} \cdot 3 = 1.$$

So the total length removed is exactly 1, which is the full length of the original interval $[0, 1]$.

From the point of view of length, we have removed *everything.*

And yet not every point has been removed.

This is the first major lesson of the Cantor set: in infinite mathematics, *having no length is not the same thing as being empty.*

That single sentence already feels like an insult to common sense.

A Set Made of Dust

It is tempting to imagine the Cantor set as a kind of infinitely delicate dust. It is not continuous like a line, nor isolated like a handful of scattered dots. It lies somewhere in between: broken, fragmented, and yet deeply structured.

The image of "dust" is useful here because it captures the peculiar tension at the heart of the Cantor set. It feels almost invisible, and yet it stubbornly refuses to disappear. Every time one expects it to collapse into nothingness, it reveals that something essential has been left behind.

So where are all these surviving points hiding? The answer is one of the most satisfying parts of the story: the Cantor set can be read directly from the way numbers are written.

The Ternary Secret

Most of us are used to writing numbers in base 10. For example,

$$0.5 = \frac{5}{10}, \qquad 0.37 = \frac{3}{10} + \frac{7}{100}.$$

But for the Cantor set, base 3 is far more revealing.

In base 3, we write

$$0.1_3 = \frac{1}{3}, \qquad 0.2_3 = \frac{2}{3}, \qquad 0.02_3 = \frac{2}{9}.$$

Now look again at the construction.

At the first step, we remove the middle third $\left(\frac{1}{3}, \frac{2}{3}\right)$. These are precisely the numbers whose ternary expansion begins with

$$0.1\cdots_3 .$$

At the second step, we remove the middle thirds of the two remaining intervals. These are exactly the numbers whose ternary expansions begin with

$$0.01\cdots_3 \quad \text{or} \quad 0.21\cdots_3\,.$$

At each stage, we remove numbers whose ternary expansion acquires the digit 1 in a certain place.

So a number survives *every* stage of the Cantor construction if and only if its ternary expansion uses only the digits 0 and 2.

That means the Cantor set consists exactly of numbers of the form

$$0.a_1a_2a_3\cdots_3$$

where each a_i is either 0 or 2.

This is a remarkably simple description of a deeply strange object.

Third Surprise: It Is Too Large to Count

Now comes the real shock.

Each point in the Cantor set corresponds to an infinite sequence of 0s and 2s:

$$(a_1, a_2, a_3, \dots), \qquad a_i \in \{0, 2\}.$$

But this is essentially the same as an infinite binary sequence

$$(b_1, b_2, b_3, \dots), \qquad b_i \in \{0, 1\},$$

because we can simply replace

$$0 \leftrightarrow 0, \qquad 2 \leftrightarrow 1.$$

So every point in the Cantor set corresponds to an infinite binary expansion.

And there are uncountably many such sequences.

This means the Cantor set has the same cardinality as the real numbers in $[0, 1]$.

So we are forced to accept something profoundly counterintuitive:

A set can have no length at all, and yet still be as large as an interval in terms of cardinality.

The Cantor set is tiny in the geometric sense, but enormous in the set-theoretic sense.

And mathematics, completely unbothered by our discomfort, insists that both statements are true at once. This is perhaps the deepest trick the Cantor set plays on us. In everyday thinking, we expect ”more” to mean longer, larger, or fuller. But the Cantor set forces us to separate these ideas. A set can be large in the sense of containing many points and yet small in the sense of occupying no measurable length. What appears, at first glance, to be almost nothing turns out to be a rich and inexhaustible infinity.

Why This Matters

The Cantor set is compelling not merely because it is strange, but because it exposes a fault line in our intuition.

In ordinary life, size feels simple. A longer stick is bigger than a shorter one. A larger box contains more than a smaller one. Geometry and quantity appear to move together.

The Cantor set tears these ideas apart.

It shows that there is more than one way for a set to be “large.” One can measure length, area, or volume. One can also count elements — or try to count them and fail. These are not the same notion of size, and infinity exposes the difference with particular ruthlessness.

What remains after the Cantor construction looks almost like nothing. But it is not nothing. It is an object that survives every attempt to erase it.

And perhaps that is what makes it so memorable. The Cantor set is not just a construction. It is a lesson in mathematical humility.

Conclusion

The Cantor set begins with an innocent instruction: remove the middle third, then repeat forever.

What emerges is one of mathematics’ most elegant paradoxes.

We remove intervals until no length remains, yet infinitely many points survive. We destroy the geometry, but not the set. We are left with something that is almost nothing, and yet far too large to count.

Infinity does not behave as finite intuition would like it to. It is not obliged to be sensible. It is not obliged to be kind.

And sometimes, when mathematics removes everything we thought mattered, it leaves behind something even more beautiful.

References

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