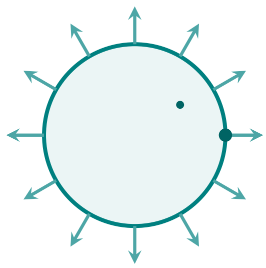
The Tragic Tale of Harriet the Hedgehog

and Why She Could Never Lie Her Hair Flat

A gentle introduction to the Hairy Ball Theorem, vector fields,

and the topology of the sphere

Christopher Henry



Harriet (attempting to look tidy)

Contents

1 Harriet Has a Bad Fur Day... **1**

2 Spines, Arrows, and Vector Fields **2**

3 The Euler Characteristic **2**

4 "Indices": How Bad is a Cowlick? **3**

5 The Poincaré–Hopf Theorem: Everything Must Balance **3**
5.1 Why Harriet is Doomed 4

6 Harriet's Smug Cousin (Terry the torus) **4**

7 There is Always a Point on Earth with NO wind at all **4**

8 Harriet Makes Peace with Mathematics **5**

1 Harriet Has a Bad Fur Day...

Meet Harriet. Harriet is a hedgehog (as the title suggests), and like all hedgehogs, her body is covered in sharp little spines. However, unlike most hedgehogs, Harriet is *deeply* persistent. Every morning, she wakes up and attempts to comb every single one of her spines so that they all lie flat against her body, pointing smoothly in one consistent direction with no bits sticking up and no bald patches.

Aaaaand, every single morning, she fails. (Poor Harriet, she just wants to look good)

No matter how carefully she combs, starting at the top, working around the sides, smoothing everything into a perfect swirl, there is always, *ALWAYS*, at least one silly spine that refuses to lie flat. Either it sticks straight up (a "**zero**": a spine pointing nowhere useful), or she runs into a horrible tuft where the spines clash and refuse to cooperate (also, ultimately, a zero in disguise).

Harriet thinks she is just bad at grooming, misfortuned by the heavens to be naturally inept at taking care of herself. But she's wrong. *It is mathematically impossible to do what she is trying to do.*

This is the "**Hairy Ball Theorem**", a silly sounding, yet cool and wonderful theorem in maths, and by the end of my silly tale of our beloved spiky animal, you will understand exactly why she is doomed, why it rains on her planet, and what on earth this has to do with the way wind blows on Earth.

2 Spines, Arrows, and Vector Fields

Before we can rescue (or rather forever disappoint...) Harriet, we need the right tools to tackle her pokey problem.

Imagine you're standing on the surface of a perfectly round ball (a sphere (S^2)). At each point (p) on the sphere, we can draw an arrow that lies flat against the surface at that point. It cannot poke into the ball or stick out away from it – it must be *tangent* to the sphere (think of basic GCSE tangents of a graph!). The direction and length of the arrow can vary from point to point, but it must ALWAYS be tangent.

> **Formal Note:**
> Now, formally, at each point $p \in S^2$, the set of all tangent arrows forms a two-dimensional (vector) space called the "**tangent space**" (T_pS^2). If we choose one arrow continuously at every point, we get a **tangent vector field** $V : S^2 \to TS^2$, a smooth assignment $p \mapsto V(p) \in T_pS^2$.

Think of Harriet's spines as exactly this: a tangent vector field on the sphere, one arrow per point on her (we assume spherical, because otherwise things become complicated) body, each arrow pointing in the direction and with the length of the corresponding spine. Harriet wants a vector field with *no zeros* – no point where the arrow has zero length, i.e. no spine that collapses to a useless dot.

The Hairy Ball Theorem says: *no such thing exists.*

> **The Hairy Ball Theorem** (Poincaré 1885; Brouwer 1912)
> Every continuous tangent vector field on the 2-sphere S^2 must vanish somewhere: there is no continuous map $V : S^2 \to TS^2$ with $V(p) \in T_pS^2$ for all p and $V(p) \neq 0$ for all p.

3 The Euler Characteristic

To understand *why* Harriet is so so so unfortunate, we need to talk about this cool number that every surface carries around: its **Euler characteristic** χ.

Take any surface and then cut it up into flat faces (polygons), connected at edges and meeting at vertices [It doesn't matter how you cut it up]. Then compute:

$$\chi = V - E + F$$

where V is the number of vertices, E the number of edges, and F the number of faces. Quite magically, this number is *always the same* no matter how you triangulate: it only depends on the

topology of the surface* , not on how you cut it [hence, the euler characteristic is a topological invariant!]

*(A simple analogy to explain what it means if the topology is the same across two shapes is whether one can be deformed into another i.e. a cube can be squeezed into a sphere)

> *Example.* Triangulate the sphere S^2 as a tetrahedron: $V = 4, E = 6, F = 4$. Then $\chi(S^2) = 4 - 6 + 4 = 2$. Try a cube instead: $V = 8, E = 12, F = 6$, giving $8 - 12 + 6 = 2$. Always 2.
> For a torus (the surface of a donut): $\chi(T^2) = 0$.

This number turns out to be the key to Harriet's pity. The bridge between χ and vector fields is one of the great theorems of topology, and this is what we will explore to finally crack down on our hedgehog's dilemma!

4 "Indices": How Bad is a Cowlick?

Suppose a vector field V on a surface has a "zero" at some point p: a point where the arrows collapse to nothing. Near p, the arrows must be flowing in *some* pattern - they might be spiralling in, rotating around p, or flowing in from one direction and out from another. The **index** of the zero at p measures how many times the vector field rotates as you walk in a small circle around p.

> To use an example, draw a small circle γ around the zero p. As you walk around γ once (counterclockwise), the angle θ that V makes with the horizontal rotates by $2\pi \cdot \text{ind}(p)$ for some integer $\text{ind}(p) \in \mathbb{Z}$. This integer is the **index** of the zero.

Common zeros and their indices:

Type of zero	**What it looks like**	**Index**
Source (arrows pointing out)	$\bullet \rightarrow$	$+1$
Sink (arrows pointing in)	$\leftarrow \bullet$	$+1$
Rotation (vortex)	arrows circling p	$+1$
Saddle (flows in, then out)	like a mountain pass	-1

5 The Poincaré–Hopf Theorem: Everything Must Balance

FINALLY! We have obtained all the keys to be able to state the theorem that makes Harriet's life a misery: The Poincaré–Hopf Theorem. It says that the indices of all the zeros of a vector field must add up to exactly $\chi(S)$, the Euler characteristic of the surface. It doesn't matter *where* the zeros are, or what the vector field looks like everywhere else - just that the indices must always sum to the same number.

> **Poincaré–Hopf Theorem**
> Let V be a smooth vector field on a compact surface S with only isolated zeros $p_1, \ldots, p_k$. Then:
> $$\sum_{i=1}^{k} \text{ind}(p_i) = \chi(S).$$

You may be wondering *"Why should this be true?"* Well, let's triangulate the surface and build a "canonical" vector field that has exactly one source (index $+1$) inside each face, one sink (index

$+1$) at each vertex, and one saddle (index -1) at the midpoint of each edge. The total index is then:

$$F \cdot (+1) + V \cdot (+1) + E \cdot (-1) = F + V - E = \chi(S).$$

Any other vector field can be continuously deformed into this one (away from the zeros), and using the fact that the index of a zero is invariant under small perturbations, it can be shown how this sum is preserved.

5.1 Why Harriet is Doomed

Now, we are truly nearing the end of our investigation. Suppose Harriet could comb her spines perfectly: a continuous tangent vector field on S^2 with *no* zeros. Then the sum of the indices over all zeros is an empty sum, equal to zero. But Poincaré–Hopf says this sum must equal $\chi(S^2) = 2$.

$$0 \neq 2. \quad \lightning$$

And now it is clear to us that Harriet cannot win - she will never be able to win. Every attempt to comb her spines will inevitably produce at least one zero: at least one point where the spines have no direction, a bald cowlick, a vortex, an unavoidable imperfection. The universe of maths will simply not allow $0 = 2$.

6 Harriet's Smug Cousin (Terry the torus)

Somewhere in the forest lives a creature called "Terry" who is topologically a torus (perhaps a very strange doughnut-shaped animal... well, let's not judge animals by their appearances) and he can comb her fur perfectly flat, with no cowlicks at all. Why?

Because $\chi(T^2) = 0$. Poincaré–Hopf now says the indices of the zeros must sum to 0. And that is perfectly achievable for the fortunate Terry: just have *no* zeros at all. The empty sum equals zero.

Concretely, the torus $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ admits the constant vector field $V(p) = (1, 0)$ for every p - just "point right everywhere" - and this descends to a nonvanishing tangent vector field on the torus. The sphere (aka Harriet) has no such luxury sadly.

> In general, a compact surface admits a nonvanishing tangent vector field iff $\chi(S) = 0$. Among the orientable surfaces, these are precisely the tori of genus (number of holes) $g \geq 1$. The sphere ($g = 0$) is the only orientable surface with $\chi = 2$, and it alone among closed orientable surfaces is hairy-ball-theorem-cursed (wow, Harriet really is unfortunate - if you don't feel bad for her now, I don't know what to say).

7 There is Always a Point on Earth with NO wind at all

Here is the most beautiful real-world consequence of Harriet's tragedy. The wind on Earth's surface is, at any given moment, a tangent vector field on the sphere: at each point on the globe, the wind has a direction and a speed. By the Hairy Ball Theorem, at every instant, there must be at least one point on Earth where the wind speed is exactly zero.

Not "nearly zero". Exactly, provably, topologically *zero*. The theorem guarantees it: the atmosphere has no choice.

Similarly, if you try to define a smooth field of *magnetic field lines* that are all tangent to the surface of a spherical magnet with no poles, you cannot. Magnetic poles on a sphere are not a design flaw - they are a mathematical inevitability.

8 Harriet Makes Peace with Mathematics

In the end, Harriet stopped trying to comb her spines. She accepted her cowlick - that one inevitable swirl on the top of her head where the spines have nowhere to go - and wore it with pride.

After all, it is not a flaw. It is a beautful feature that Harriet has (aka a really cool theorem!).

Every sphere, everywhere in the universe, in any dimension you like (well, every *even*-dimensional sphere - the odd-dimensional ones escape the theorem), carries this same inescapable mark.

The Earth has a windless point. The magnetic field has poles. The hedgehog has a cowlick.

They all, in their own way, are honouring a constraint so deep it is not a law of physics but a law of *mathematics itself*: you cannot tile the sphere with arrows. The topology will not allow it. And $2 \neq 0$, no matter how hard you, or the globe, or Harriet combs.

How I learnt all this: Lee's smooth manifolds, MIT opencourseware notes, and Algebraic Topology by Hatcher!
...well only a bit, the rest is too scary for Harriet and I :)