The Gambler's Dilemma: When Strategy Matters and When it Cannot

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1.1 Introduction

Picture the scene: a drunk man leaves his house after a party for some fresh air. However, he is faced with a dilemma: the street outside his house extends infinitely left and right. He stumbles left and right with equal probability at each step. Will he ever return home?

Perhaps at first, you would think that eventually, the man would keep ending up closer to one of ends of the street, which would make returning home an extremely unlikely event. In fact, the probability that the man returns to home is exactly 1. No matter how far he stumbles, he returns home. We will return to prove this, but first we must understand the mathematical structure behind his stumbling, which turns out to be identical to a simple betting game.

1.2 Gambler's ruin

Suppose two gamblers, A and B, are playing a game. A starts with £i and B starts with £(N-i), respectively. They flip a coin between them, and if heads, then A takes £1 from B, and vice versa. This is repeated until one of the gamblers is ruined and left with £0. At each step, the wealth of A moves up or down with equal probability, same as the drunkard moves left or right. This is the mathematical process known as a random walk.

And what if the coin is biased against A? What if the coin is in A's favour? Does strategy matter then, and by how much? The answer to these questions will divide all possible games into three categories, the boundary of which is one of the most powerful theorems in probability theory.

Before diving in, it is worth posing the question in its most general form. Suppose now that each player can decide how much to bet and when to stop playing, can a stopping strategy combined with smart bet sizing guarantee profit in a fair game? The scope of gambler's ruin is too narrow to answer this; we are only using it as our starting point to build the tools we need to answer the general question.

2.1 Solving gambler's ruin

Let P(i) = P(player A ruined | A starts at £i), where A wins a flip with probability p and loses with probability q = 1-p. The two extreme cases, which we will call the boundary conditions are:
P(0) = 1, since if A gets to £0 then A is certain to be ruined.

P(N) = 0, since if A wins all the money, then the game ends and they can't be ruined.

Let the current wealth of A be represented by a particle on a number line, starting at position i. On any flip, A moves to i+1 with probability p, or i-1 with probability q. Therefore, by the Law Of Total Probability, we obtain the equation:

$P(i) = p\cdot P(i+1) + q\cdot P(i-1)$

Rearranging:

$p\cdot P(i+1) - P(i) + q\cdot P(i-1) = 0$

Since (p + q) = 1, we can write P(i) = (p+q)P(i)

$\implies p\cdot P(i+1) - (p+q)\cdot P(i) + q\cdot P(i-1) = 0$

$\implies p[P(i+1) - P(i)] = q[P(i) - P(i-1)]$

Now we will use a substitution to form difference equations. Let D(i) = P(i) - P(i-1), which is the difference between consecutive terms. The equation now becomes:

$p\cdot D(i+1) = q\cdot D(i)$

$\implies D(i+1) = \frac{q}{p} D(i)$, which is a geometric sequence.

Let $r = \frac{q}{p}$, Then:

$D(2) = r\cdot D(1)$
$D(3) = r\cdot D(2) = r^2\cdot D(1)$

$\implies D(i) = r^{(i-1)}\cdot D(1)$

D(1) + D(2) + ... + D(i) = P(i) - P(0) (Since the sum telescopes)
==> P(i) = P(0) + D(1) + D(2) + ... + D(i)
==> P(i) = 1+ D(1)· [1 + r + r^2 + ... + r^(i-1)]

When r ≠ 1, summing the geometric series:

P(i) = 1 + D(1)· $\frac{1-r^i}{1-r}$

==> P(N) = 1 + D(1)· $\frac{1-r^N}{1-r}$

Applying the second boundary condition P(N) = 0:

0 = 1 + D(1)· $\frac{1-r^N}{1-r}$

==> D(1) = $\frac{-(1-r)}{1-r^N}$

Substituting back:

P(i) = 1 + $[\frac{-(1-r)}{1-r^N}]\cdot(\frac{1-r^i}{1-r})$

==> P(i) = $\frac{r^i-r^N}{1-r^N}$

But this is the probability of ruin, so P(A survives | A starts with wealth £i) = 1-$\frac{r^i-r^N}{1-r^N}$= W(i)

==> W(i) = $\frac{1-r^i}{1-r^N}$

==> W(i) = $\frac{1-\left(\frac{q}{p}\right)^i}{1-\left(\frac{q}{p}\right)^N}$, since r = $\frac{q}{p}$

2.2 Implications of the formula

Suppose that A is betting against a casino with infinite wealth, as N → ∞ , three cases emerge:

If p< 0.5, r > 1: $r^N \to \infty$, so W(i)→ 0. Therefore, unfavourable games against infinite wealth guarantees ruin.

If p = 0.5, r = 1: We get W(i) = $\frac{0}{0}$ so we need to apply L'Hôpital's rule:

Numerator:$\frac{d}{dr}(1 - r^i) = -i \cdot r^{i-1}$

Denominator:$\frac{d}{dr}(1-r^N)$= $-N \cdot r^{N-1}$

Taking the limit as $r \to 1$

W(i) =$\frac{i}{N}$, so as N→ ∞ , W(i)→ 0. Ruin is still certain, even with a fair game, but this time only because of the wealth asymmetry.

If p > 0.5, r<1: $r^N \to 0$, so W(i)→ 1-$\left(\frac{q}{p}\right)^i$>0. There is now a genuine possibility of survival, even against infinite wealth.

P = 0.5 is the precise boundary between guaranteed loss and possible survival. Additionally, there is no trace of strategy at all in the formula, since there is no freedom to stop or change bet size. The real question is whether freeing up the game changes anything, and this requires us to understand the geometry of paths in a random walk.

3.1 Paths of a random walk

We have already solved gambler's ruin algebraically, but by analysing the paths of a random walk, we can arrive at results algebra alone can't reach, as well as the answer to whether the drunkard makes it home.

Circling back to the drunkard's walk, let the man's position over time be represented as a particle on a cartesian grid, with position along the road on the y-axis and time intervals on the x-axis. The particle starts at (0, a). At each movement, the particle moves 1 unit along the x-axis and either 1 or –1 along the y-axis.

3.2 The reflection principle

Call the set of paths that start at a, touch 0 at some point and end up at position b(where b>0). Call this set of paths set A. On figure 1, a path in set A touches 0 at time F and ends up at b, reflect the path up to F about the x axis, leaving the path after F unchanged. The reflected path that starts at –a also ends up crossing the x axis and ending up at b. Call the set of paths from –a to b set B.

Every path in set A maps onto a path in set B due to the initial reflection. Every path in set B also maps onto set A, since any path from -a to b must cross 0 at some point, we can find its first crossing, reflect everything before it, and recover a unique path in set A.

Therefore, we have bijection between set A and set B, meaning elements in each set map 1 to 1 to an element in another set.

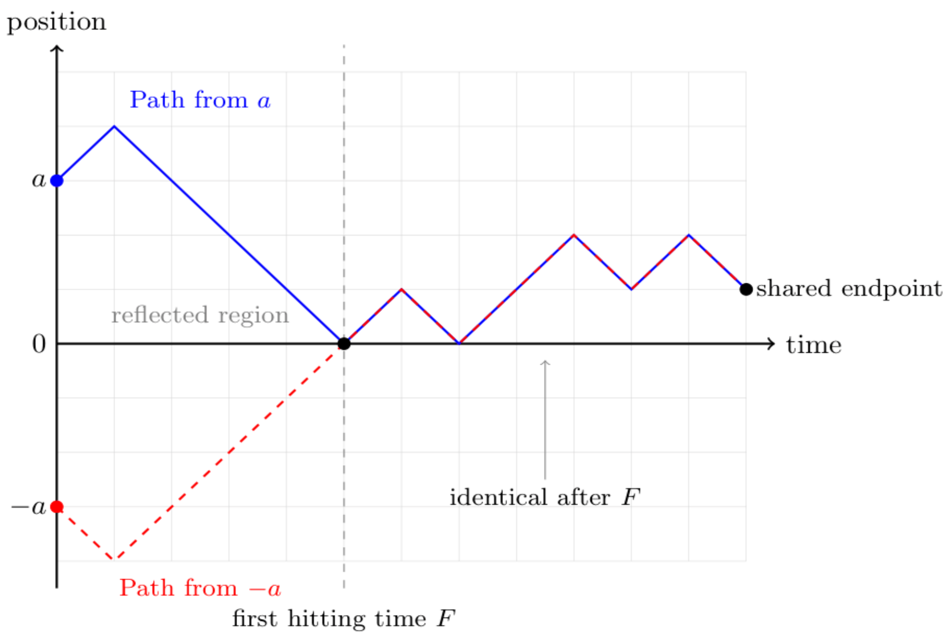

Figure 1

==> P(path from A touches 0 ∩   ends at b) = P(unrestricted path from –a ends at b)

We will examine this later when assessing how stopping time can be used in games when stopping time is not fixed.

3.3 Proving the drunkard returns home

First, note that it requires 2n steps for the man to return to 0, since it requires n left steps and n right steps, in some order.

Therefore, P(man returns to 0) is the number of ways to arrange n left and right steps divided by the total number of paths of length 2n, which is 2^{2n}.

==> P(returns after 2n steps) = $\frac{2nCn}{4^n}$

For a large n, we can use Stirling's approximation [1], which estimates n! ≈ $\sqrt{2\pi n} \cdot \left(\frac{n}{e}\right)^n$

==> $\frac{2nCn}{4^n} \approx \frac{1}{\sqrt{\pi n}}$

E[total returns] = P(return at step 2) + P(return at step 4) + P(return at step 6) + …

==> E[total returns] = $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}}$

Since $\frac{1}{\sqrt{\pi n}} = \frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}}$ and $\frac{1}{\sqrt{n}} \geq \frac{1}{n}$ for all n≥1, we obtain:

$$\frac{1}{\sqrt{\pi}} \cdot \frac{1}{\sqrt{n}} \geq \frac{1}{\sqrt{\pi}} \cdot \frac{1}{n}$$

Since the harmonic series $\frac{1}{n}$ diverges, so by comparision $\frac{1}{\sqrt{\pi}} \cdot \sum_{n=1}^{\infty} \frac{1}{n}$, also diverges

$\therefore$ $\sum_{n=1}^{\infty} \frac{1}{\sqrt{\pi n}} = \infty$ since it diverges. So expected returns to the origin = ∞

==> P(returns to origin) = 1

It is interesting to note that, for a 2D walk, P(return) is still 1. However, starting with 3D, walks are no longer guaranteed to return home, but proving this is beyond the scope of this paper. This has led to the saying "A drunk man will find his way home, but a drunk bird may get lost forever." [2]

4.1 Martingales- tying it all together

Formally, a martingale is a stochastic(random) process X_0 ,X_1...satisfying:

$E(X_{n+1}|X_1,X_2, \ldots, X_n) = X_n$

This means that the best predictor for future value of the random variable X, given all the values of X up to now, is exactly your current wealth. This process has no exploitable structure.

For the gambler and the drunkard with p = 0.5:

$E(X_{n+1}|X_n) = \frac{1}{2}\ (X_n+1)+\frac{1}{2}\ (X_n-1) = X_n$

==> Fits a martingale

For p≠0.5, $E(X_{n+1}|X_n) = p(X_n+1) + q(X_n-1) = X_n(p+q) + (p-q) = X_n+(p-q)$

If p>0.5(biased for us), value drifts upwards; this is a submartingale.

If p<0.5(biased against us), value drifts downwards; this is a supermartingale.

4.2 The Optional stopping theorem

Now, we will consider whether allowing the gambler to stop at any time will bring in strategy.

First, we must set out some conditions a “reasonable” stopping time. [3]

1. The game can't involve reaching infinite wealth.

2. The game can’t take infinite time.

3. The game has $E[T]<\infty$ and no huge jumps in bet sizing. This ensures the game is still “fair”. If one of the conditions applies, then OST can be used.

The OST suggests that $E[S_T] = E[S_0] = K$, where S_i = wealth after i rounds, T is the stopping round and K is the starting value.

For gambler’s ruin, we cannot impose condition 1, as wealth can theoretically increase forever. We also cannot impose condition 2, since there is a positive probability that at a large time(C), the game hasn't ended yet. So, we must use condition 3. Since the bet size is constant at 1, we just need to check $E[T]<\infty$.

We first define a stopped martingale S_i^T, which behaves the same as a normal martingale until it stops at T, then the value remains constant.

$\therefore\ S_i^T = S_i$ for $i \leq T$

Since $0 \leq S_i^T \leq N$, this new martingale is bounded, satisfying condition 1. By the OST:

$E[S_T] = E[S_0^T] = E[S_0\ ] = K$

$\therefore$ For a fair game as defined by the OST, stopping strategies do not affect the martingale property. This formalises what the reflection principle showed geometrically, stopping at a hitting time doesn’t change expected wealth.

The famous martingale betting strategy—doubling bets each round to guarantee an expectation of 1, fails to fall under the OST. It fails under the constraints of bounded time, bounded wealth and stable progression of bet sizes. So be my guest to try the strategy, if you have an infinite debit card.

Optimising bet size- the Kelly criterion

If A bets fraction f of current wealth each round, log growth per step:

$G(f) = p \cdot \log(1+bf) \ + \ q \cdot \log(1-f)$, $b=\frac{profit}{stake}$, p = P(win)

Setting $\frac{dG}{df} = 0$ and rearranging gives the optimal fraction:

$K = p-\frac{q}{b}$ [4]

This is the Kelly criterion, which yields the optimal bet fraction per round to maximise long term growth, balancing growth and chance of ruin, since it penalises ruin($\log(0)=-\infty$).

If $P(win) \leq 0.5$, the criterion would suggest betting ≤ 0

==> If $P(win)<0.5$, survival time is maximised with minimal bet sizes.

Therefore, we have shown that $p=0.5$ is the knife-edge separating games where strategy is vital and ones where they are useless.

Conclusion

As steps become smaller, the random walk converges to Brownian motion, inheriting the martingale property. This is the foundation of modern financial markets and the Efficient Market Hypothesis [5]. Brownian motion also models natural processes like particle diffusion [6]. The gambler and the trader are both governed by the question whether clever strategies can beat a random process, the answer to which is determined strictly by mathematics.

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