Is Math Logical?

Prologue:

Before the play unfolds, we must understand one crucial term called “logic”. Logic is “the science of thinking about or explaining the reason for something using formal methods.” This essentially means that logic is a framework that governs whether a statement is true or false. In a mathematical sense, it is the study of reasoning and proof. An example of a logical expression is “if A implies B, and A is true, B must also be true”. Similarly, this logical expression feels closely related to arithmetic, like how $1 + 1 = 2$ can be described from the definitions of 1,+,=, and 2. Perhaps this means that math and logic are identical, and the proof of this equivalence is left as an exercise for the reader. Whether that’s true, false, or completely made up, you’ll find out as soon as you read on!

Cast of Characters:

Gottlob Frege (1848 - 1925): The Visionary

Bertrand Russell (1872 - 1972): The Disrupter

Alfred North Whitehead (1861 - 1947): The Scaffolding

Kurt Gödel (1906 - 1978): The Destroyer

Act 1:

Immanuel Kant - “Mathematical knowledge is the knowledge gained by reason from the construction of concepts.”

Gottlob Frege was born in 1848 in a small town in Germany, where he spent the majority of his career at the University of Jena. He represents the stereotypical image of a genius who was largely ignored by his contemporaries and in no way attempted to seek any fame. He sought something far more profound, certainty within Mathematics. His vision was to strip every theorem to its bones and prove that what remained was logic, and that mathematics is just logic in disguise, forming the idea of “logicism.”

His first major work was called Begriffsschrift, roughly translated to “concept-writing.” This was the introduction of the formal language notation for logic, though many of Frege’s notations have been changed in comparison with the modern notation. Another piece of his work that established the foundation of “logicism” is Die Grundlagen der Arithmetik, translated to “The Foundations of Arithmetic," where he laid out his argument that numbers are not physical objects nor are mental constructions, but rather they are logical objects, defined by rules. Within the book, Frege also criticises Kant on his belief that numerical statements are synthetic a priori, like how 7 + 5 = 12. What does synthetic a priori mean, you may ask? A priori means to know independently of experience; for instance, you do not need to go outside and count the number of trees to know that it's there, while synthetic means to add new information. This means that “7 + 5 = 12” tells you something that is not already contained in the definition of the terms “7” and “5.” So for Kant, numerical operations are certain like logic, but require intuition to be understood. More precisely, you have to almost picture it. Frege rejects this concept entirely, proposing that arithmetic is not synthetic; it is analytic. By unpacking the definition of “7” and “5” carefully enough, you can arrive at the conclusion that 7 + 5 = 12 is already contained within the concept itself. There is no need for intuition or picturing; pure logic is enough. Therefore, it can be seen that Kant’s synthetic a priori is the philosophical obstacle that Frege’s logicism aims to demolish.

To prove that arithmetic could be reduced to simple logic, Frege’s approach was to define mathematical objects simply through logical properties, where in today’s world we would call them “sets”. And to answer the question of what a number is, Frege’s answer would be that numbers are sets. Frege’s method of deducing that numbers are sets is proven within his book, Die Grundlagen der Arithmetik, where he was able to build numbers from “nothing” by utilising the logic of sets. For instance, he defined the number 0 by considering that there exists a set that is not identical to itself $\chi \neq \chi$. But since every object in the universe is identical to itself, there are zero objects that fall into this set, creating the number 0.

Frege’s work drew the attention of many mathematicians across Europe. One of them was Bertrand Russel, and he had a letter to write.

Act 2:

Dear Frege,

I have been thoroughly studying through your work, Die Grundlagen der Arithmetik, and I could not agree more with all of your points. However, I encountered one difficulty.

Imagine a village populated only with grown men with beards, with a peculiar law against growing beards. Specifically, the law states that the village barber must shave all and only those men of the village who do not shave themselves. But the barber himself lives in the village too, and he's a man. So does he shave himself?

Tell me your thoughts on this.

Very Respectfully Yours,
Bertrand Russell

On June 16, 1902, the day Frege received the two-page letter, he was devastated. It physically and emotionally broke him. His life's work was disproved by a paradox.

At the time, Frege utilised "Naive Set Theory" to prove his idea of logicism. However, the system itself was fundamentally flawed and was built on intuition rather than being defined strictly. Its rules were simple and were just an extension of these two ideas:

1. Unrestricted composition, meaning that we can make any sets that we want

2. Set identity is determined by membership, meaning that what makes a set what it is is just what is inside of it

These rules seem obvious. But Russel found something hidden beneath the first rule of unrestricted composition. According to the idea, any set can be formed; therefore, we can create sets that contain themselves or any other sets. For example, we can construct the set of all sets that don't contain themselves. If it does contain itself, then it is in the set itself, but the only way to be in the set is to meet the condition of not being in the set. If it doesn't contain itself, then it meets the condition and should be in the set by definition. Ultimately, both possibilities will reach a paradox.

This contradiction implies that Naive set theory does not work. The scaffolding of logicism is flawed.

After the discovery of this paradox, Russel approached Alfred North Whitehead and began working on the solution, partly out of guilt, partly out of obsession. Out of their joint effort, they published *Principia Mathematica* in 1910, 1912, and 1913. It was dense, roughly two thousand pages long, and was created with the sole purpose of reconstructing the foundation of mathematics with extreme precision. It was renowned for requiring 379 pages of proof, only then could the authors finally collect all of the tools required to formally prove that $1 + 1 = 2$.

The Principia Mathematica was a monument. But some of the axioms felt like assumptions, and not genuine, pure logic. The dream of reducing mathematics to logic was looking increasingly less viable.

And then came Gödel.

Act 3:

As Russell and Whitehead believed they had salvaged their dream, a 25-year-old Austrian Mathematician named Kurt Gödel published a result that forces us to change our view on mathematics. A result that devastated the field of logicism, simply proving them wrong. In 1931, Gödel published his first "Incompleteness Theorem" proving that in any formal system powerful enough to perform arithmetic, there will be true statements which cannot be proven. You can think of it like a dictionary, which defines words by other words inside it. Gödel proved that in such a system, there will always be certain true statements, which can not be proven by its own rules. Gödel's second incompleteness theorem went even further and showed that any consistent formal system cannot prove its own consistency. This implies that there is no guarantee and no demonstration that shows that mathematics does not contain contradictions.

If mathematics cannot prove its truth using its own rules, then logic alone can never serve as the foundation of mathematics. Through this, Gödel revealed that no matter how carefully you build a system, there will inevitably be truths that slip unnoticed, making mathematics unable to be completely reducible to logic.

With Gödel's incompleteness theorem, the dream was over. Frege's ambition of obtaining certainty has failed. Russell and Whitehead's attempt to recover that dream is ultimately destroyed. The dream of showing that mathematics is nothing more than pure logic remained forever …

Incomplete.

Epilogue:

Frege envisioned the dream. Russell destroyed it. Whitehead helped to pick up the pieces. Gödel placed the final nail into the coffin of logicism. Mathematics is not reducible to logic. Yet Frege's effort was not wasted, the pursuit of the dream served as the foundation of modern logic, and everything that subsequently followed was built on the premise Frege had offered.

After all, the proof was never left as an exercise to the reader. Rather, it was a journey in which the readers learned something they weren't expecting.

Bibliography:

- "Logic_1 Noun - Definition, Pictures, Pronunciation and Usage Notes | Oxford Advanced Learner's Dictionary at OxfordLearnersDictionaries.com." *Oxfordlearnersdictionaries.com*, 2023, www.oxfordlearnersdictionaries.com/definition/english/logic_1.
- Kaplan, Jeffrey. "Russell's Paradox - a Simple Explanation of a Profound Problem." *YouTube*, 8 Sept. 2022, www.youtube.com/watch?v=ymGt7I4Yn3k. Accessed 24 Feb. 2023.
- Wikipedia Contributors. "Logicism." *Wikipedia*, Wikimedia Foundation, 30 Oct. 2025, en.wikipedia.org/wiki/Logicism.
- "Immanuel Kant Quote: "Philosophical Knowledge Is the Knowledge Gained by Reason from Concepts ; Mathematical Knowledge Is the Knowledge..."" *Quotefancy.com*, 2026, quotefancy.com/quote/832045/Immanuel-Kant-Philosophical-knowledge-is-the-knowledge-gained-by-reason-from-concepts. Accessed 12 Apr. 2026.
- Klement, Kevin C. *Frege, Gottlob | Internet Encyclopedia of Philosophy*. iep.utm.edu/frege/.
- Irvine, Andrew David. "Bertrand Russell." *Stanford.edu*, 7 Dec. 1995, plato.stanford.edu/archives/win2015/entries/russell/#RWL. Accessed 11 Apr. 2026.
- "Begriffsschrift." *Wikipedia*, 18 June 2023, en.wikipedia.org/wiki/Begriffsschrift.

- ---. “The Foundations of Arithmetic.” *Wikipedia*, Wikimedia Foundation, 16 Nov. 2025, en.wikipedia.org/wiki/The_Foundations_of_Arithmetic.

- Wikipedia Contributors. “Analytic–Synthetic Distinction.” *Wikipedia*, Wikimedia Foundation, 4 May 2022, en.wikipedia.org/wiki/Analytic%E2%80%93synthetic_distinction.
- www.thatmarcusfamily.org/philosophy/Course_Websites/Readings/Russell%20-%20Letter%20to%20Frege.pdf.

- “Naive Set Theory.” *Wikipedia*, 22 May 2023, en.wikipedia.org/wiki/Naive_set_theory.
- Irvine, Andrew David, and Harry Deutsch. “Russell’s Paradox.” *Stanford.edu*, 8 Dec. 1995, plato.stanford.edu/archives/win2014/entries/russell-paradox/. Accessed 11 Apr. 2026.
- *Barber Paradox*. www.oxfordreference.com/display/10.1093/oi/authority.20110803095446216.
- ---. “Principia Mathematica.” *Wikipedia*, Wikimedia Foundation, 9 Apr. 2019, en.wikipedia.org/wiki/Principia_Mathematica.
- ---. “Kurt Gödel.” *Wikipedia*, Wikimedia Foundation, 19 July 2020, en.wikipedia.org/wiki/Kurt_G%C3%B6del.
- Raatikainen, Panu. “Gödel’s Incompleteness Theorems (Stanford Encyclopedia of Philosophy).” *Stanford.edu*, 11 Nov. 2013, plato.stanford.edu/entries/goedel-incompleteness/.
- “Self-Reference.” *Wikipedia*, 6 Oct. 2023, en.wikipedia.org/wiki/Self-reference.