The Dirichlet Integral: How to Integrate without an Antiderivative

When solving a definite integral, we all follow the same formula. Find the antiderivative, plug in the limits and we get the answer. But what if a function does not have an antiderivative? For the vast majority of functions, this would be a dead end. But for the Dirichlet Integral, this is not the case.

The Dirichlet Integral

But first, why do we even care about it? The Dirichlet integral is the integral of the unnormalised sinc function from 0 to infinity,

$$\int_0^\infty \frac{\sin x}{x} dx$$

and is used in our everyday lives. From converting noise into precise, mathematical waves that speakers play to being used in MRI and CT scans, it really is everywhere you look around.

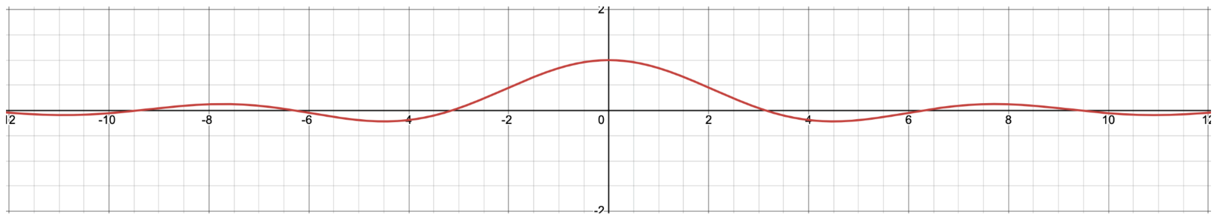
Here are some properties of $\frac{\sin x}{x}$ we should note:

(we will refer to $\frac{\sin x}{x}$ as $f(x)$ from now)

1. $\lim_{x \to 0} \frac{\sin x}{x}$ is 1 (which can be proved easily using L'Hôpital's rule). Despite being divided by x, the function does not blow up to infinity at any point. (but is still not defined at $x = 0$)
2. It is an even function, as $f(-x) = \frac{-\sin x}{-x} = f(x)$. This will be used later on when we instead find $\int_{-\infty}^{\infty} f(x)dx$ and halve the result.
3. $\int_0^\infty f(x)dx$ is convergent (otherwise this would be quite a short essay). Using integration by parts: $\int_0^\infty \frac{\sin x}{x} dx = \int_0^1 \frac{\sin x}{x} dx + [\frac{1}{x}(-\cos x)]_1^\infty - \int_1^\infty \frac{\cos x}{x^2} dx$

Because $\frac{\sin x}{x}$ does not blow up near $x = 0$, $\int_0^1 \frac{\sin x}{x} dx$ is definitely convergent, and the square bracket evaluates to $\cos(1)$, which is also a finite number. Lastly, $\frac{\cos x}{x^2} < \frac{1}{x^2}$, so $\int_1^\infty \frac{\cos x}{x^2} dx < \int_1^\infty \frac{1}{x^2} dx$. $\int_1^\infty \frac{1}{x^2} dx = 1$, so we know $\int_1^\infty \frac{\cos x}{x^2} dx < 1$ and is definitely finite as well.

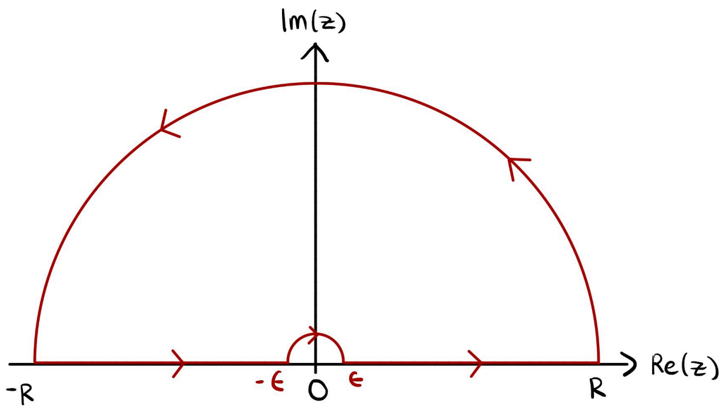
All this can be seen when $f(x)$ is plotted:



Contour Integration

Let's begin with the most difficult (and my favourite). For this method we will call $I = \int_{-\infty}^{\infty} \frac{\sin x}{x} dx$, so $\frac{1}{2} I = \int_0^\infty \frac{\sin x}{x} dx$. We will now set $g(z) = \frac{e^{iz}}{z}$, so $f(x) = \mathrm{Im}(g(z))$, which can be seen using Euler's formula that $e^{iz} = \cos(z) + i\sin(z)$. Thus we now have $I = \mathrm{Im}(\int_{-\infty}^{\infty} \frac{e^{iz}}{z} dz)$.

When we integrate along the real line, there is only one path between the limits; a straight line. For complex numbers, we are in a 2D plane; hence, there are an infinite number of ways to get from one point to the next. This means that we can create a function to go between limits, which we call contours. We will now create a contour that will allow us to use Cauchy's theorem (more on this later). The contour (see below) is a semi-circle of radius $\lim_{R \to \infty}$ with an indented semi-circle of radius $\lim_{\epsilon \to 0}$. It is important to note that complex integration does not involve areas like it does with real integration.



(Notice the arrows; the direction of the contour matters)

We will then apply Cauchy's Integral Theorem, which states that if a complex function $g(z)$ is holomorphic (complex differentiable) at every point on and inside a simple closed contour C (a closed loop that does not intersect itself) in a simply connected domain (essentially does not have holes), then the contour integral around C is zero. This can be written as

$$\oint_C g(z)dz = 0$$

(The circle shows the contour is closed)

where C represents the closed contour. Using the quotient rule on $g(z)$, we get that:

$$g'(z) = \frac{z(ie^{iz})-(e^{iz})(1)}{z^2}$$

So $g'(z)$ clearly exists for all z except for $z = 0$.
We can now see why we chose this contour shape, so that it will 'hop' over the point where $g(z)$ is not holomorphic due to the division by 0, allowing us to apply this theorem. We can now split the integral:

$$\oint_C g(z)\, dz = \int_{-R}^{-\epsilon} g(z)dz + \int_{\epsilon}^{R} g(z)dz + \int_{C_\epsilon} g(z)dz + \int_{C_R} g(z)dz = 0$$

where C_ϵ and C_R represents the small indented semi-circle arc and the larger semi-circle arc respectively.

Notice that as we take the $\lim_{\epsilon \to 0}$, and that the $\lim_{R \to \infty} \int_{-R}^{-\epsilon} g(z)dz + \int_{\epsilon}^{R} g(z)dz$ tends to cover the real line axis, and can then be written as $\int_{-\infty}^{\infty} g(z)dz$.

In order to evaluate $\int\limits_{C_R} g(z)dz$, we will use Jordan's lemma, which essentially just says that integrals like this where the 'dome' extends to infinity is equal to 0. We first use the substitution $z = Re^{i\theta}$, where $e^{i\theta}$ represents a unit circle in the complex plane (easily seen through Euler's formula), and R represents the radius. Using the chain rule and applying the limits such that we get the semi-circle we want, we get:

$$\int\limits_{C_R} \frac{e^{iz}}{z} dz = \int\limits_0^{\pi} \frac{e^{i(Re^{i\theta})}}{Re^{i\theta}} (iRe^{i\theta}) d\theta$$

(For the 'dome', we go counter-clockwise from R to -R, which is known as the 'positive' direction, so we will end up with a non-negative answer)

$$= i\int\limits_0^{\pi} e^{iR(\cos\theta + i\sin\theta)} d\theta$$

$$= i\int\limits_0^{\pi} e^{iR\cos\theta - R\sin\theta} d\theta$$

$$J = i\int\limits_0^{\pi} e^{iR\cos\theta} e^{-R\sin\theta} d\theta$$

We use the fact that $|\int g(z)dz| \leq \int |g(z)||dz|$ (summing the magnitude of individual values is greater than the magnitude of their 'net' value) and that $|ab| = |a||b|$ inside the integral. This leaves us with:

$$|J| \leq |i|\int\limits_0^{\pi} |e^{iR\cos\theta}||e^{-R\sin\theta}| d\theta$$

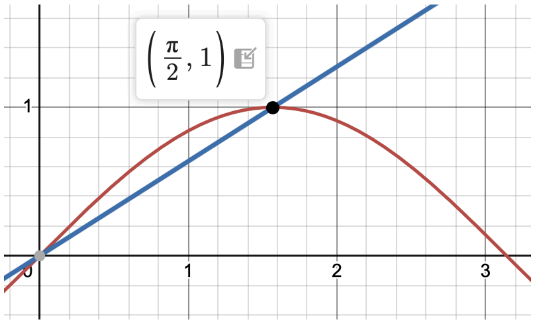
($|i|$ is just 1, and $e^{iR\cos\theta}$ is a point on the unit circle and has modulus one)

$$|J| \leq \int\limits_0^{\pi} |e^{-R\sin\theta}| d\theta$$

Because $\sin\theta$ is symmetrical about $\frac{\pi}{2}$, we can half the upper limit and multiply the value of the integral by 2:

$$|J| \leq 2\int\limits_0^{\frac{\pi}{2}} |e^{-R\sin\theta}| d\theta$$

This is difficult to integrate, so we can instead replace $\sin\theta$ with $\frac{2\theta}{\pi}$, as $\sin\theta \geq \frac{2\theta}{\pi}$ when $0 \leq \theta \leq \frac{\pi}{2}$. This will not only make it easy to integrate but will also cancel other terms later on.



(graph of $\sin\theta$ against $\frac{2\theta}{\pi}$)

This implies that $e^{-R\sin\theta} \leq e^{-R\frac{2\theta}{\pi}}$, so we can replace $e^{-\sin\theta}$ in our integral:

$$|J| \leq 2\int_0^{\frac{\pi}{2}} |e^{-R\sin\theta}| d\theta \leq 2\int_0^{\frac{\pi}{2}} |e^{-R\frac{2\theta}{\pi}}| d\theta$$

$$|J| \leq 2\int_0^{\frac{\pi}{2}} |e^{-R\frac{2\theta}{\pi}}| d\theta$$

$$\leq 2\left[\frac{-\pi}{2R}e^{-R\frac{2\theta}{\pi}}\right]_0^{\frac{\pi}{2}}$$

$$\leq \frac{-\pi}{R}(e^{-R} - 1)$$

$$\leq \lim_{R\to\infty} \frac{-\pi}{R}(e^{-R} - 1)$$

$$|J| \leq 0$$

Because the modulus of something can't be less than 0, $|J| = 0$ so $J = 0$.

Now all we have left to do is find the value of $\int_{C_\epsilon} g(z)dz$. First, we write $g(z)$ as its Taylor series expansion. (For now we will keep it general.)

$$g(z) = \frac{a_{-1}}{(z)} + a_0 + a_1(z) + a_2(z)^2 + \ldots$$

When we try to integrate $g(z)$, we can instead integrate each term of the Taylor series instead, which we will show is equal to 0 for all terms but one.

$$\int_{C_\epsilon} a_n(z)^n dz$$

Let $z = re^{i\theta}$

(as z goes around a circle of radius r around the origin)

$$= a_n \int_{\pi}^{0} (re^{i\theta})^n ire^{i\theta} d\theta$$

$$= ir^{n+1} \int_{\pi}^{0} e^{i(n+1)\theta} d\theta$$

(The limits are 0 and π respectively as we are going clockwise)

$$= ir^{n+1}\left[\frac{1}{i(n+1)} e^{i(n+1)\theta}\right]_{\pi}^{0}$$

$$= \frac{r^{n+1}}{n+1}((1) - (-1)^{n+1})$$

$$= \frac{r^{n+1}}{n+1}(2 \ or\ 0)$$

As we take $\lim_{r \to 0}$, both tend to zero. This, however, assumes that n is not -1, as we would get a division by 0. We now need to account for the case that $n = -1$.

$$\int_{C_\epsilon} \frac{a_{-1}}{(z)} dz$$

$$= \int_{\pi}^{0} \frac{a_{-1}}{re^{i\theta}} (ire^{i\theta}) d\theta$$

$$= \int_{\pi}^{0} a_{-1} i d\theta$$

$$= -a_{-1} i\pi$$

We know the value of a_{-1} as we know the coefficients of the Taylor expansion:

$$e^{iz} = 1 + iz - \frac{z^2}{2!} - \frac{iz^3}{3!} ...$$

$$\frac{e^{iz}}{z} = \frac{1}{z} + i - \frac{z}{2!} - \frac{iz^2}{3!} ...$$

So from this we can see that our coefficient of $(z)^{-1}$ is just 1, so $a_{-1} = 1$,

meaning that $\int_{C_\epsilon} g(z)dz = -i\pi$. (This is essentially how the Fractional Residue theorem works.)

Now that we know the value of each integral, we can bring everything together. Earlier we showed:

$$\oint_C g(z)\, dz = \int_{-R}^{-\epsilon} g(z)dz + \int_{\epsilon}^{R} g(z)dz + \int_{C_\epsilon} g(z)dz + \int_{C_R} g(z)dz = 0$$

We now know that:

1. $\int_{-R}^{-\epsilon} g(z)dz + \int_{\epsilon}^{R} g(z)dz = \int_{-\infty}^{\infty} g(z)dz$

2. $\int_{C_\epsilon} g(z)dz =- i\pi$

3. $\int_{C_R} g(z)dz = 0$

Therefore:

$$\int_{-\infty}^{\infty} g(z)dz - i\pi + 0 = 0$$

$$\int_{-\infty}^{\infty} g(z)dz = i\pi$$

We know that $I = \mathrm{Im}(\int_{-\infty}^{\infty} g(z)dz)$, so:

$$I = \pi$$

We want to find $\frac{1}{2}I$, so we can simply halve the result. Finally:

$$\int_{0}^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Laplace Transformation

Next, we will try to replicate the result with Laplace transforms. When we apply the Laplace transform to a function $f(t)$, it transforms the function to:

$$L\{f(t)\} = \int_{0}^{\infty} e^{-st} f(t)dt$$

Where s is a positive constant so that the integral always converges. This will help us simplify our integral into a form where we can apply the inverse Laplace transform, giving us our answer.

We first need to define our integral as a function of t, as Laplace transforms integrate with respect to t. We let:

$$I(t) = \int_{0}^{\infty} \frac{\sin(tx)}{x} dx$$

We are not actually changing our function; we will simply set $t = 1$ at the end to ensure we get the answer we are looking for. We place t inside the sine argument so it will cancel other terms later on and simplify the integrand. Now apply the Laplace transform to $I(t)$:

$$L\{I(t)\} = \int_0^\infty e^{-st} I(t)dt$$

$$= \int_0^\infty e^{-st} \left(\int_0^\infty \frac{\sin tx}{x} dx\right) dt$$

$$\int_0^\infty \int_0^\infty e^{-st} \frac{\sin tx}{x} dx dt$$

Swapping the order of integration (i.e. swapping dt and dx) would allow us to move $\frac{1}{x}$, which would allow us to integrate the inner integral easily. To see if this is valid, we would typically use Fubini's theorem, which says that if the integral of the absolute value of the integrand is finite, the order of integration does not matter. However, the value of $\int_0^\infty |\frac{\sin tx}{x}| dx$ diverges to infinity, so the absolute volume of our double integral does not actually converge. Note that Fubini's theorem give sufficient but not necessary conditions, meaning the swap could possibly still be valid without meeting them. We will carry out the swap and see if we run into any problems.

$$\int_0^\infty \int_0^\infty e^{-st} \frac{\sin tx}{x} dx dt = \int_0^\infty \int_0^\infty e^{-st} \frac{\sin tx}{x} dt dx$$

$$= \int_0^\infty \frac{1}{x} \int_0^\infty e^{-st} \sin(tx)\, dt dx$$

Notice $\int_0^\infty e^{-st} \sin(tx)\, dt$, is the Laplace transform of $\sin(tx)$, so we can rewrite it as:

$$= \lim_{R \to \infty} \int_0^R \frac{1}{x} L\{\sin tx\} dx.$$

We will now change our focus to finding what $L\{\sin tx\}$ is. First we establish what the Laplace transform of a derivative is.

$$L\{f'(t)\} = \int_0^\infty e^{-st} f'(t)dt$$

$$= [e^{-st} f(t)]_0^\infty - \int_0^\infty f(t)(-se^{-st})dt$$

(for a Laplace transform to exist is must grow slower than the decay from e^{-st}, so we can assume that $[e^{-st} f(t)] = 0$ when we plug in $\lim_{t \to \infty}$)

$$= (0 - f(0) + s\int_0^\infty e^{-st} f(t)dt$$

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Now we will find the Laplace transform of a second derivative by substituting $f''(t) = g'(t)$.

$$L\{g'(t)\} = sL\{g(t)\} - g(0)$$
$$L\{f''(t)\} = sL\{f'(t)\} - f'(0)$$
$$L\{f''(t)\} = s^2 L\{f(t)\} - sf(0) - f'(0)$$

This is useful for $\sin x$, as the second derivative of it is the negative of itself.
Let $f(t) = \sin(tx)$:

$$L\{-x^2 \sin(tx)\} = s^2 L\{\sin(tx)\} - (x)$$

Using the linear property of the Laplace transform for the terms that don't have t in it:

$$-x^2 L\{\sin xt\} = s^2 L\{\sin xt\} - x$$
$$L\{\sin xt\}(-x^2 - s^2) = -x$$
$$L\{\sin xt\} = \frac{-x}{-x^2 - s^2}$$
$$= \frac{x}{x^2 + s^2}$$

Now we substitute this into our previous equation:

$$\lim_{R \to \infty} \int_0^R \frac{1}{x}\left(\frac{x}{x^2+s^2}\right)dx$$
$$= \lim_{R \to \infty} \int_0^R \frac{1}{x^2+s^2}dx$$
$$= \lim_{R \to \infty} \left[\frac{1}{s}\arctan\left(\frac{x}{s}\right)\right]_0^R$$
$$L\{I(t)\} = \frac{1}{s}\left(\frac{\pi}{2} - 0\right)$$
$$L^{-1}\{L\{I(t)\}\} = \frac{\pi}{2}L^{-1}\left\{\frac{1}{s}\right\}$$

We can use Lerch's theorem which states if $L\{f(t)\} = L\{g(t)\}$, then $f(t) = g(t)$ for all $t \geq 0$. This means if we can find a function that, when the Laplace transform is applied to it becomes $\frac{1}{s}$, will be equal to $L^{-1}\{\frac{1}{s}\}$. The Laplace transform of 1 seems like it might be equal to $\frac{1}{s}$:

$$\int_0^\infty e^{-st}(1)dt = \frac{-1}{s}\left[e^{-st}\right]_0^\infty$$
$$= \frac{-1}{s}(0 - 1)$$
$$= \frac{1}{s}$$

This means $L^{-1}\{\frac{1}{s}\} = 1$, and finally we get:

$$I(t) = \frac{\pi}{2}$$

Since the value of $I(t)$ does not depend on t, $I = \frac{\pi}{2}$.

It is interesting to see that we get the correct value here despite our integral not meeting Fubini's conditions, so this integral would be one of the 'lucky' ones that don't meet the sufficient condition but still work.

Leibniz Integral Rule:

Last but definitely not least, we have the Leibniz Integral Rule (or Feynman's trick). In general, to use the Leibniz Integral Rule, we begin by introducing an integral function $I(a)$, which will allow us to differentiate with respect to a under the integral sign.

Similar to the Laplace transform, we need to add a term with a in it that will be easy to integrate and will make the entire integral converge. One function that meets all these criteria (if you couldn't already guess it) is the exponential function, specifically:

$$e^{-ax}$$

We also add x to the exponent, which will cancel out the denominator of $f(x)$ when we differentiate it. Now define $I(a)$ and differentiate both sides with respect to a:

$$I(a) = \int_0^\infty \frac{\sin x}{x} e^{-ax} dx$$

$$\frac{d}{da} I(a) = \frac{d}{da} \int_0^\infty \frac{\sin x}{x} e^{-ax} dx$$

We will now move the derivative on the RHS to be inside the integral. Intuitively, you are finding the rate of change of the total sum of $\frac{\sin x}{x} e^{-ax}$, which seems like it would be equal to summing the individual rates of change of $\frac{\sin x}{x} e^{-ax}$. However, because we are integrating to infinity, which often causes problems, we need to ensure our function is 'well-behaved' as we do this for all x.

For this to be a legal move, we will use the Dominated Convergence Theorem, which states that if a function is bounded in absolute value by a single curve with a finite total area, you may swap the limit and the integral ($\lim \int = \int \lim$).

Because differentiation is a limit, if we let $f(x, a) = \frac{\sin x}{x} e^{-ax}$, then:

$\frac{d}{da}\int_0^\infty f(x, a)dx = \lim_{h \to 0} \int_0^\infty \frac{f(x,a+h)-f(x,a)}{h}dx$. Hence, we need to find a 'roof' function that is greater than $\lim_{h \to 0} \frac{f(x,a+h)-f(x,a)}{h}$ for all values, while having a finite area to show that we can move the differentiation into the integral. Because x does not depend on a, we can instead take the partial derivative (only differentiating with respect to one variable, treating the other as a constant)

$$\lim_{h \to 0} \frac{f(x,a+h)-f(x,a)}{h} = \frac{\partial}{\partial a}\left(\frac{\sin x}{x}e^{-ax}\right)$$

$$\frac{\partial}{\partial a}\left(\frac{\sin x}{x}e^{-ax}\right) = - sin(x)e^{-ax}$$

$$| - sin(x)e^{-ax}| \le e^{-ax}$$

$$\int_0^\infty | - sin(x)e^{-ax}|dx \le \int_0^\infty e^{-ax}dx$$

Because we have a 'roof' function that is clearly finite, we can safely move the derivative through the integral sign as it obeys the Dominated Convergence Theorem.

$$\frac{d}{da}I(a) = \int_0^\infty \frac{\partial}{\partial a}\frac{\sin x}{x}e^{-ax}dx$$

$$= \int_0^\infty \frac{\sin x}{x}(- x)e^{-ax}dx$$

$$J = - \int \sin(x)e^{-ax}dx$$

We will now find the value of J using integration by parts twice.

$$- J = - e^{-ax}\cos x - \int(- \cos x)(- ae^{-ax})dx$$

$$= - e^{-ax}\cos x - \int_0^\infty \cos(x)ae^{-ax}dx$$

$$=-e^{-ax}\cos x-(\sin(x)ae^{-ax}-\int_0^\infty sin(x)(-a^2e^{-ax})dx)$$

$$-J=-e^{-ax}\cos x-a\sin(x)e^{-ax}+a^2J$$

$$J(-1-a^2)=-e^{-ax}\cos x-a\sin(x)e^{-ax}$$

$$J=\frac{e^{-ax}\cos(x)+ae^{-ax}\sin(x)}{a^2+1}$$

Now plug in the limits:

$$\frac{d}{da}I(a)=[J]_0^\infty$$

$$\frac{d}{da}I(a)=\frac{1}{a^2+1}[e^{-ax}\cos(x)+ae^{-ax}\sin(x)]_{x=0}^{x=\infty}$$

$$=\frac{1}{a^2+1}(0-(1+0))$$

$$\frac{d}{da}I(a)=\frac{-1}{a^2+1}$$

$$I(a)=\int\frac{-1}{a^2+1}da$$

$$=-\arctan(a)+C$$

To find the value of C, we can simply plug in a to be $\lim_{a\to\infty}$ so that $I(a)$ collapses to 0:

$$I(a)=\int_0^\infty\frac{\sin x}{x}e^{-ax}dx$$

$$\lim_{a\to\infty}I(a)=0$$

Plug this into a previous equation:

$$0=-\lim_{a\to\infty}\arctan(a)+C$$

$$0=-\frac{\pi}{2}+C$$

$$C=\frac{\pi}{2}$$

$$I(a)=-\arctan(a)+\frac{\pi}{2}$$

Because our original function does not have e^{-ax}, we can simply set a to be 0, so e^{-ax} becomes 1, leaving us with

$$I(0) = \int_0^{\infty} \frac{\sin x}{x}(1)dx$$

$$\int_0^{\infty} \frac{\sin x}{x} dx = - \arctan(0) + \frac{\pi}{2}$$

$$= \frac{\pi}{2}$$

Final Remarks

Now that we have found the same answer from 3 completely different methods, we can view the result as the beautiful harmony in math or just an *extremely* elaborate way to check our answer. I hope this essay has taught you as much as it has taught me and perhaps inspired you to find 3 more ways to solve this integral.

Sources:

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