TRM 2026 Math Contest Entry

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1 Introduction

As with many things in my life, the idea for this paper came to me when I was supposed to be doing something else. More accurately it came to me while I was working on a PSet based upon a lecture from the morning before on the random walker problem. The game itself is simple enough to understand, but if I've done my job right you'll soon see is anything but. The game is as follows: Suppose you have an n by m grid.

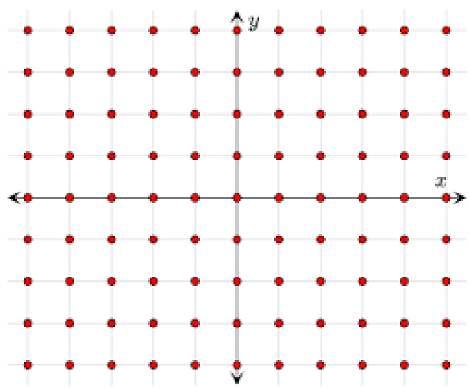

Figure 1:

The first player (A) starts by connecting two adjacent lattice points with a line. The second player (B) then connects another two lattice points. Play continues until a loop is formed, that is a certain area of the board is completely enclosed by lines. It is of great import to note that once a single closed loop has been made the game instantly ends and the player who completed the loop loses. Now this game is interesting but it has yet to get at the heart of why I love this topic so much, which is for the same reason I love the random walker problem so much, there are many questions you can ask (and answer) about it. On brand with my love of combinatorial game theory, the question is

1) What (if any) is the optimal strategy to win the game?

2 Investigation

For this section I will be unable to feign as though we are discovering this together because the truth is that I know where we are going, I have a map if you will, and figured it out some time ago. There is an optimal strategy and it's simpler than one may think (which makes it extra beautiful).

The first and most important thing to notice about this game is that it is a impartial game, which is a fancy way to say that the available moves do not depend on which player is playing. To give an example suppose you had two players L and R, for Louis(he/him) and Rita (she/her), and some board state S. Now give S to L, he should now be able to make a move. Now you give the same S to R, if she could make all of the same moves that L could make, then the game is impartial.

So now since both players have all the same moves the winner is determined by the turn order. You'll quickly find this to be true precisely because of the fact that our game is impartial. Consider that you have some board state and the players play it out to completion (with perfect accuracy) and L wins. If you gave that same position to R, she would win, and not only that she would win in the exact same way. This is not necessarily true for a game that is not impartial (partisan) such as chess, as L could have a checkmate in one and even if R could move instead of L chances are it wouldn't matter.

Let's now partition our moves into 2 categories moves that lose and moves that instantly don't lose. Since we can assume that Louis and Rita are reasonable players at the game, two things will happen

1. The number of total available moves will decrease, as a L or R just made one.

2. The number of total safe available moves will decrease as one was just taken (and that move will sometimes take away adjacent edges).

This reduces the game to a game in which each player starts taking away safe moves where whoever takes away the last safe move wins.

WE HAVE NOW ARRIVED AT A PARTICULARLY BEAUTIFUL LANDMARK. We have just reduced our game to another game called Nim. In fact all impartial games can be reduced to Nim or a Nim-like game (for some certain edge cases), this is exactly the Sprague-Grundy Theorem, perhaps the most unusually complete theorem in game theory, as not only does it tell you that a game has a solution it tells you exactly what your game can be simplified to.

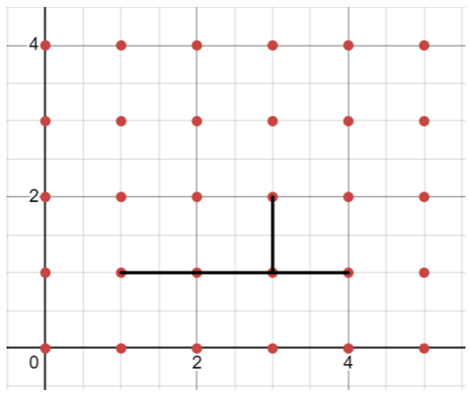

Figure 2: A Forest with 26 trees

Now you may be thinking ”If this theorem existed the entire time, why didn’t we just use it to jump right to this point”, but where’s the fun in that?

Now that we know that there is doomsday clock on our game, we had better figure out when it will ring. To do this we need to introduce some more machinery, namely trees and forests. Trees are graphs, which are comprised of nodes and edges (*See Figure 2*).These graphs cannot have a closed loop and cannot have multiple edges between the same two nodes. They also require all of the nodes in the tree to be connected, *See Figure 2*. A forest is a collection of trees, so if you include a lone node with the tree in figure 2 that would be a forest of two trees or of size two (trees can very importantly be a single node).

If my explanation was good enough you should be able to see that out starting position (*See Figure 1*) is a forest of size $n * m$. Now L plays, once he plays he must connect two nodes thus combining two trees into one, and therefore decreasing the size of the forest by one. Thus when we will eventually combine all trees into one tree. When we reach this last tree it must be a spanning tree, meaning it contains all nodes. However when we add another edge to this graph it must create a closed loop as since there is already one way to get to every vertex by adding in another edge it will create new pathway through nodes A and B and thus you can get from any node to be either taking the new edge or not taking the new edge, thusly creating a loop.
Another way to think about it is that now that we have a spanning tree our graph is connected and thusly we can use the Euler characteristic. The Euler characteristic is an equation with states that for all connected shapes $V(vertices) - E(edges) + F(faces) = 2$ So for our game state $V = nm$, $E = nm - 1$ and $F = 1$ for the plane that we draw our graph on.

$$nm - (nm - 1) + 1 = 1 + 1 = 2$$

However if we add in another edge and since the number of vertices stays constant we must have another faces and that face is the region enclosed by the loop we just formed.

Since we cannot pick add an edge after we have a single tree, which is after $nm - 1$ turns as that is the number of trees you need to chop down, and consequently the game will last $nm - 1$ turns. And therefore you can choose first or second based upon the parity (oddness or evenness) of nm. THE GAME WAS ALWAYS DETERMINISTIC.

3 Final Remarks

I have hopefully convinced you that the game's determinism is not only completely inevitable and baked into the game's structure, but actually the coolest part of the game. If there are any mathematical errors please let me know as I solved the game myself and I'm worried I've made a mistake. This game is also not over, as in the future I would also like to investigate this game as an area of graph theory where all graphs have to obey the restriction set by checker tile adjacency.

THANK YOU SO MUCH FOR READING!

4 References

[1] Sprague, R. P. (1935). Über mathematische Kampfspiele. Tōhoku Mathematical Journal, 41, 438–444.

[2] Grundy, P. M. (1939). Mathematics and games. Eureka, 2, 6–8.