Fractals, Foliage and the Key to the Universe

Introduction

Mathematics is beautiful. You and I both know this - from nature's Fibonacci spirals to solving calculus problems on a whiteboard, maths has a certain, unique, unmistakable beauty. Today I am going to be talking about what, in my opinion, is the most beautiful area of maths - fractals.

So what even is a Fractal?

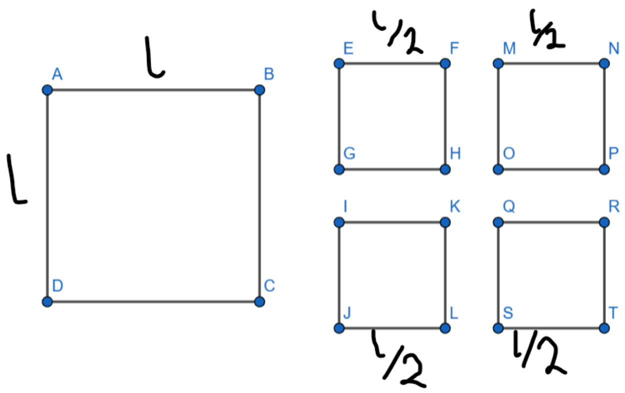
The simple definition of a fraction is a shape with a non-integer dimension. But what does this mean? How can a shape have a non-integer dimension? Well, this is where we need to be more precise with our definition of what a dimension is, and we'll look at 2 different types of dimension - topological, and Hausdorff (When did maths get so confusing?).

First, let's define what a topological dimension is.

Topological dimensions are: "The number of coordinates needed to specify the position of a point in space, where the dimension of a space is determined by the highest dimension of its subspaces." Ok, that's not too bad, maybe just a little bit confusing, but for the purposes of this essay we can think about it in simple, intuitive terms - length = 1 dimension, area = 2 dimensions, volume = 3 dimensions, and I'll stop there because I do not want to get into higher dimensional geometry today. All right, got that? Let's move on.

Now, before jumping into the world of fractals, let's start with some familiar shapes, just to make sure we understand what's going on.

Take a square with a length of l.
Splitting the square into 4 smaller squares, each with side lengths l/2, we can see that as we scale down the square by a factor of ½, the area is ¼ of the original, and then it doesn't take a genius to work out that the area scale factor is the length scale factor squared.



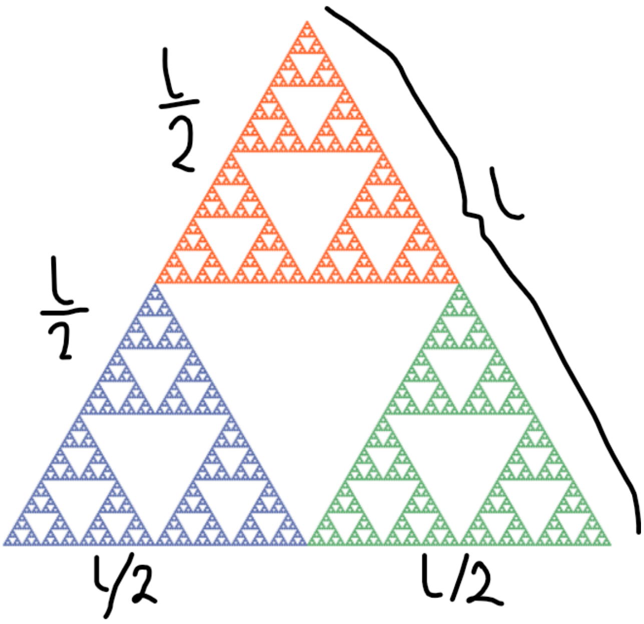
(Please excuse my awful handwriting)

$\frac{1}{2}$ ^n = $\frac{1}{4}$
n(log($\frac{1}{2}$)) = log($\frac{1}{4}$)
n = log($\frac{1}{4}$) ÷ log($\frac{1}{2}$) = 2

Clearly, the square has a topological dimension of 2, so what is it that we just calculated? You guessed it, it's the Hausdorff dimension! So what is the Hausdorff dimension? It is a measure of the roughness or irregularity of a set or object, and how that scales with the object. Calculating the Hausdorff dimension of a cube can be done in a similar way, as a cube can be split into 8 smaller cubes each with side length half that of the original, so the Hausdorff dimension of a cube is 3.

Now, let's do this for one of my personal favourite fractals, the Sierpinski triangle. As it has no area, we can intuitively imagine that the triangle is made from a thin wire, and this time we are focusing on the mass of the triangle.

Here, the side length of the triangle has again been scaled with scale factor $\frac{1}{2}$, however, we have made 3 identical copies of the original triangle, each with $\frac{1}{3}$ the mass of the original. We can use the same logarithmic method as before to calculate the Hausdorff dimension.



$\frac{1}{2}$^n = $\frac{1}{3}$
n(log($\frac{1}{2}$)) = log($\frac{1}{3}$)
n = log($\frac{1}{3}$) ÷ log($\frac{1}{2}$) ≈ 1.585

Et Viola! A shape with a non-integer Hausdorff dimension! That wasn't so difficult, was it?

Now that we are armed with knowledge of Hausdorff dimensions, I think we can update our definition of a fractal:

“Any shape whose Hausdorff dimension is greater than its topological dimension” - Benoit Mandelbrot.

Keep in mind that the topological dimension of the Sierpinski triangle is actually 1 - it is made up of line segments, and has an area of 0. As 1.585 > 1, the Sierpinski triangle classifies as a fractal, whereas the square and cube, whose topological dimensions = their Hausdorff dimensions, do not.

A World of Fractals

Now that we've got some of the basics down, let's take a look at how fractals appear in our world. A common example is coastlines, so let's take a look at a very familiar island.

Pictured on the left is the island of Great Britain, and according to the Ordnance Survey the coastline is 17800km. Simple, right? But wait, according to the British Cartographic Society the coastline is 31368km, and the CIA World Factbook has it recorded as 12429km, and that's including Northern Ireland. So what is going on?

This is known as the coastline paradox, which says that the coastline of a country or island has no defined value, as if you measured with smaller "rulers", then the measured coastline will increase, as you can measure more of the nooks and crannies in the land.

Just looking at Great Britain, you can see that the coast is extremely jagged, particularly in Scotland, so larger rulers wouldn't be able to pick up all the details, which decreases the measured value of the coastline. In fact, if you had a ruler of Planck Length, which is 1.616 x 10^-35 metres, the smallest possible length, you could pick up detail at the subatomic scale, and the length of the coastline would be virtually infinite. The only problem with this, is that such a ruler would be very easy to lose.

This coastline paradox arises from the fractal nature of Earth's landmasses, which is created by marine and aerial erosion processes. So how would we go about calculating the Hausdorff dimension of Great Britain? Our previous method of dividing the length by 2 and seeing how many identical copies are made seems a bit impractical here, and this is due to a property called self similarity.

Self similarity is when an object looks "roughly the same when viewed at different degrees of magnification", and this is a property exhibited by some fractals, for example the Sierpinski triangle. However, many fractals that appear in nature do not have this property, and this unfortunately can make calculating the Hausdorff dimension more difficult.

Hausdorff Dimensions Demystified (for real this time)

Unfortunately, computing the Hausdorff dimension of a fractal is extremely difficult, especially for real coastlines, so we are going to be approximating it instead.

The method involves measuring the coastline of the island using different length rulers, then plotting a graph of the logarithm of the length measured against the length of the ruler used, to get a linear relationship.

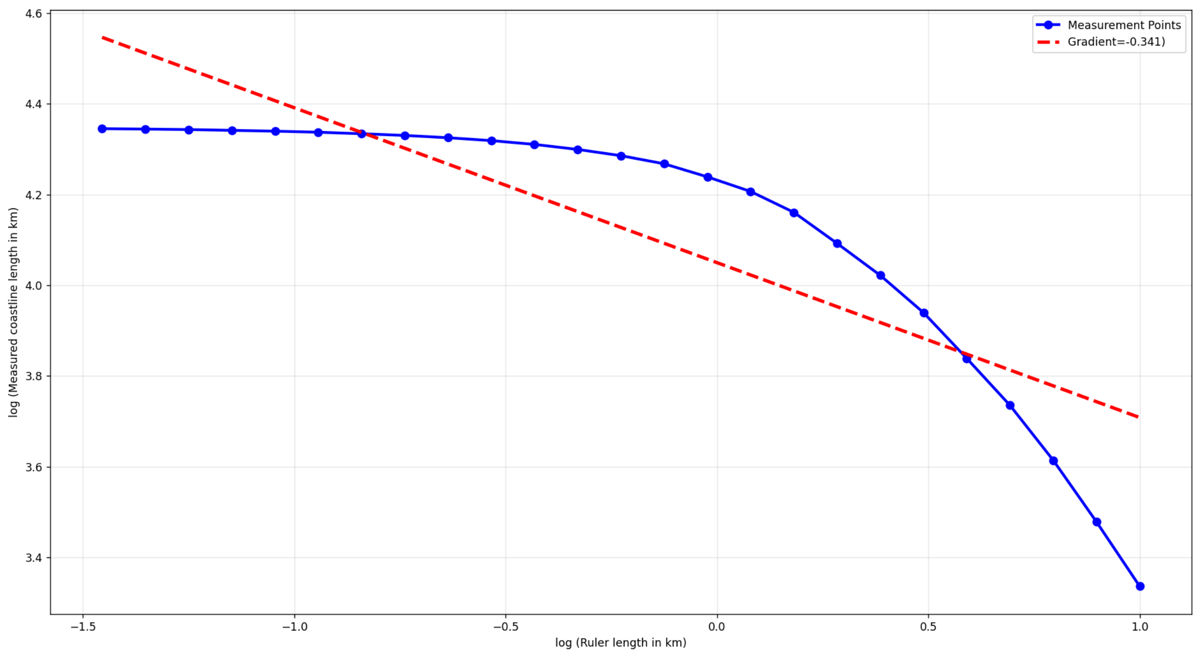
This works because, taking the measured length as L and the ruler length as l, the relationship should be approximately

L = K * l^(1-D)

Where K is a constant and D is the Hausdorff dimension. We can see that if D was 1, a perfectly smooth line, then the length just equals K, no matter the ruler length. With some basic manipulation we get:

log(L) = log(K*l^(1-D))
log(L) = log(K) + (1-D)log(l)

Which is in a familiar y=mx+c straight line graph form, with 1-D being the gradient of the line. Using a large map and some string to represent the rulers, I was able to compile numerous data points, and put them into a python program to output the graph of log(L) against log(l), and I used linear regression to create a best fit line to calculate the gradient.



From this graph we can see that the gradient of the useful data points is -0.341, so D = 1 - -0.341 which = 1.341.

Benoit Mandelbrot also used this method to estimate the Hausdorff dimension of Great Britain in his 1967 paper: “How Long Is the Coast of Britain? Statistical Self-Similarity and Fractional Dimension”, and his value was roughly 1.25, so my value isn’t too far off, given that my method may have been unorthodox. I mean, did you really expect me to walk around the coast of Britain with a 10km ruler?

There are other methods for estimating Hausdorff dimension, such as the box counting method, in which you place the fractal over a grid, count the number of squares touched by the outline of the fractal, then scale up the fractal by some factor, and then again count the number of squares touched by the outline.

Repeating this method with increasing scaling factors, you could plot a graph that approximately fits y = kx^n, where n is the Hausdorff dimension, and k is a proportionality constant.

3Blue1Brown demonstrates this concept in his video titled: “Fractals are not typically self similar”. I, however, have reached the limit of my knowledge of Python, so let’s move on.

They’re in my food?!

Believe it or not, fractals actually rear their heads in the realm of biology. First, we’ll look at everyone’s favourite vegetable, the humble broccoli.

Pictured here are 2 types of broccoli. The first is the broccoli that is most familiar - and as you can see each smaller stem is a copy of the entire thing, meaning it is self similar.

On the right is the Romanesco broccoli, which grows in Italy. Each of the spiral structures is a copy of the plant, and they all also create a natural Fibonacci spiral.

Another example of a plant that exhibits self-similar fractal properties is the fern.

Each of the smaller leaves is a copy of the whole leaf, and this pattern continues for all the branches and sub branches of the leaf.

The reason that fractals are so prevalent in plants is all to do with surface area. Plants use photosynthesis to produce glucose, and to do this energy from the Sun is required. So, to maximise the rate of photosynthesis, plants must increase the surface area of their leaves to catch as much sunlight as possible, and these fractal shapes do exactly that, while not requiring massive amounts of energy to grow, which larger leaves would.

A Fractal Future

Fractals are absolutely everywhere in nature. It's as if God decided to play a prank on us curious humans by making everything these shapes that we cannot hope to comprehend. So why do we even bother trying with them?

New research in fractals has very wide reaching implications, from the medical to the technological to the philosophical. For example, healthy human blood vessel cells tend to grow in a regular fractal pattern, and this makes detecting cancerous cells, which grow in an irregular fashion, easier.

Furthermore, fractals are beginning to be used in encryption algorithms, as their chaotic properties can enhance key sensitivity, improve the robustness of security, and decrease latency.

Fractals are also being used by researchers to help visualise chaotic systems, which could help to map potential future states, useful in areas such as economics, where it has been discovered that exchange rates have fractal properties.

Finally, the theory that interests me the most personally, is called Fractal Cosmology, which is a minority theory. The cosmological principle of the universe states that the universe is homogenous and isotropic, essentially meaning that it looks the same everywhere in all directions, at large enough scales. Fractal Cosmology, however, opposes this, and proposes that the universe itself is one giant fractal. I mean, if everything from cells in our bodies to the great galaxies of outer space are fractals, then why not the universe? This is an unpopular theorem as it does oppose our current observations, but some scientists believe that fractals may be the key to understanding our universe.

Conclusion

I'm going to keep this short and sweet, and not just because I can see the 2000 word limit approaching. I hope that you enjoyed taking this tour of fractals, and I hope that the next time you step outside you take a moment to appreciate the beauty, mystery and complexity of this fractal world that we live in.

Thank you for reading.

References

https://taylorandfrancis.com/knowledge/Engineering_and_technology/Engineering_support_and_special_topics/Hausdorff_dimension/

https://taylorandfrancis.com/knowledge/Engineering_and_technology/Engineering_support_and_special_topics/Topological_dimension/

https://touchingthelight.co.uk/coast/challenge/

https://www.sciencedirect.com/topics/mathematics/self-similarity

http://li.mit.edu/Stuff/CNSE/Paper/Mandelbrot67Science.pdf

https://www.nature.com/articles/s41598-025-04700-2

https://www.sciencedirect.com/science/article/abs/pii/S104244319900030X