

Untying the Knots

by Arush Upadhye

1. Introduction

You might be reading the title of my essay and feel completely lost about what on earth this essay is going to be about. Just bear with me. Take a second to grab the nearest pair of headphones, earphones, or even just a regular charger.

Now look at it.

Congratulations – you are staring at one of the most deceptively profound objects in all of mathematics. We live in a world of knots. It's what holds our shoes together, it's what keeps our boats moored, and also extremely good at inconveniencing headphone users. But here is a simple question for you. Perhaps too simple. And the question is: *how do you know when two knots are actually different?*

Not just different looking, but fundamentally different. In the same way that a square and a circle are different shapes, not just the same shape drawn at different scales. So now I ask you. Say you were given a long string, extremely, terrifyingly tangled, where both ends are glued together. How would you prove to me that no amount of sliding, pushing or rearranging could ever produce a plain loop? In fact, how would you even begin? Instinctively, you'd probably furiously try to untangle it. But in doing so, it would probably lead to further knotting, of not only your string, but also your brain. Or you could always cut the string and then untangle it, but where's the fun in that. So why can't we explain this failure? What was missing? Based on the fact that this is a mathematics essay, yes, you guessed correctly, the answer is algebra. With this immensely powerful tool, we could now produce an irrefutable verdict without trying a single move.

2. What Even Is a Knot?

This question is more important than we think, as it sets up a crucial rule. Knots aren't just used for tying up your shoelaces; in maths we can define it as a closed loop in a 3D space, where the two ends are glued together. *This* is a knot. Crucially, the ends are stuck, and the string may never be cut. This means that the only movement which can occur is rearrangement, pushing and sliding of the loop in space, without letting it pass through itself.

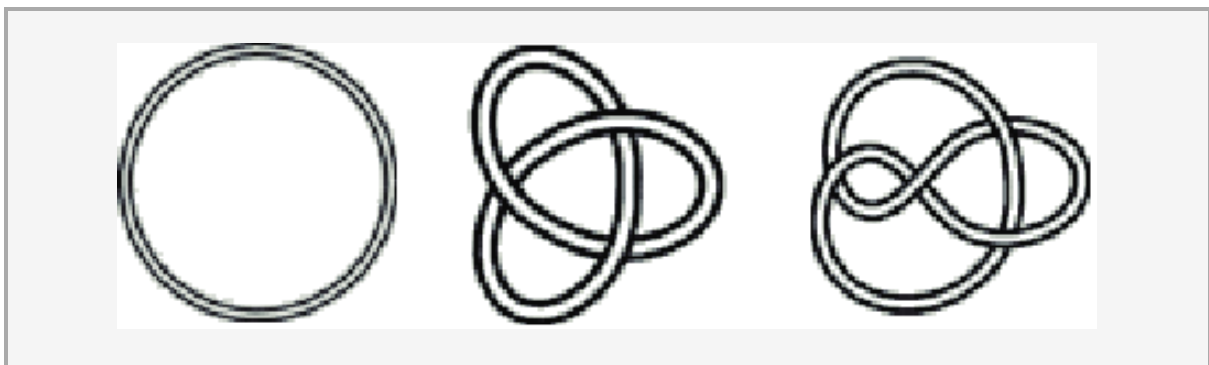


Figure 1: The three simplest knots. The unknot has no crossings, the trefoil has three, and the figure-eight has four. Are any two of them secretly the same knot?

The simplest knot is the unknot. Shockingly, it is a knot which is not a knot, but mathematically is a knot, since it fits the definition of a knot that the ends must meet. (*please excuse the tongue twister!*) The trefoil is the first actual knot, which has three crossings, as you can see in the Figure 1 above. Lastly comes the figure-eight knot, with four crossings. As you can imagine, the complexity of these knots increases as you have more crossings.

So, to reiterate the central question: the trefoil *looks* knotted. The unknot *looks* plain. But looks are not proofs. Is it possible that with sufficiently clever rearranging, no cutting, just sliding, you could transform the trefoil into the unknot? And if not, how would you prove it?

3. The Naïve Approach: Why Counting Crossings Fails

The most obvious idea is to count crossings. If the unknot has zero and the trefoil has three, would this not mean that the number of crossings correspond to a different knot? No. Of course it's not that simple. And it's because the drawing of a knot is rather meticulous. If done carelessly, one of the knots identified above could have around seven or eight different crossings.

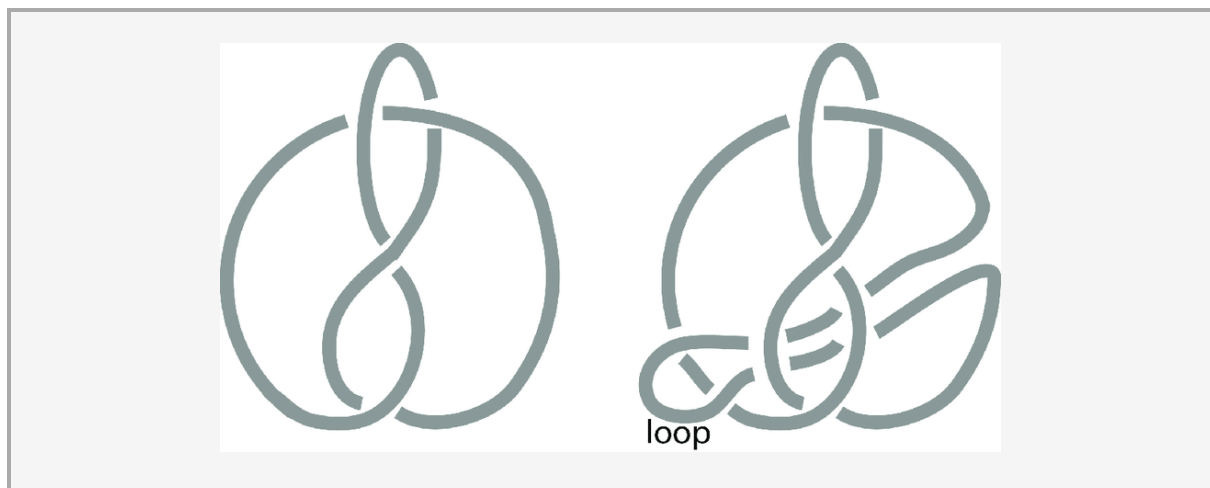


Figure 2: The same figure-eight knot drawn two ways. Crossing count alone is an unreliable identifier — the same knot can look completely different depending on how you arrange it.

Mathematicians, therefore, defined three base operations you can perform on a knot diagram without changing the underlying knot. These are known as the three Reidemeister moves (*which I will call R moves in the rest of the essay*).

- 1) Twist or untwist in either direction.
- 2) Slide one loop completely over another.
- 3) Slide a string over or under a crossing.

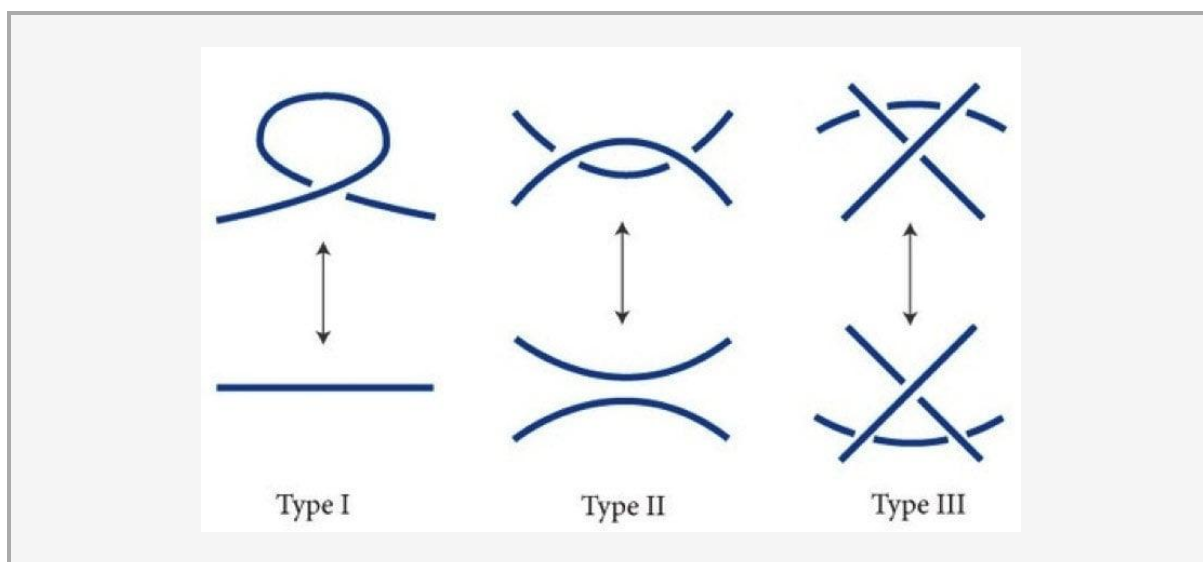


Figure 3: The three Reidemeister moves. Any legal rearrangement of a knot diagram can be broken down into some sequence of these three operations.

Essentially, what Reidemeister proved was that two knot diagrams were the same type of knot *if and only if* one could be obtained from the other by performing a sequence of these three moves. For example, if you wanted to show that a trefoil knot is the same as an unknot, you would need to perform a series of R moves on the trefoil knot, and you would need the right sequence.

However, as always, there's a catch. There is no limit on how many moves might be required to solve it. It could be ten moves, or it could be ten million. But either way, there are so many different possibilities that there is no way of knowing if you were actually going to untangle it, or if you had just given up too soon. The probability of finding the right solution to untangle the trefoil knot is so low that we can conclude that the 'naïve' approach was never going to work.

4. Invariants

The genius idea that mathematicians came up with was invariants. An invariant is a property of a knot that does not change no matter how you deform or redraw it. This means that no matter how many R moves you apply, no matter how tangled or elegant your diagram, the invariant remains fixed. This is powerful: if two knots have different invariants, they must be different knots – no further argument needed.

Here's a beautiful analogy (*created by me, which is why it's so good*) which might help. It's like a passport. If you are the same person, you would have the same passport number, regardless of changes in your appearance. Even though you look different, doesn't mean you are actually a different person as your passport number proves, but if the passport number is different, then the person is in fact entirely different.

The first great invariant accessible to us is called tricolourability.

5. Tricolourability

A strand in a knot diagram will be a piece of the string that goes from one undercrossing to the next. If we try to colour every strand of the knot using exactly three colours, we must follow one key condition:

At each crossing, the three incident strands are either all the same colour or all different colours.

That is the entirety of the rule. The question is simply: can you do it, using all three colours at least once each?

Let us start with the unknot.

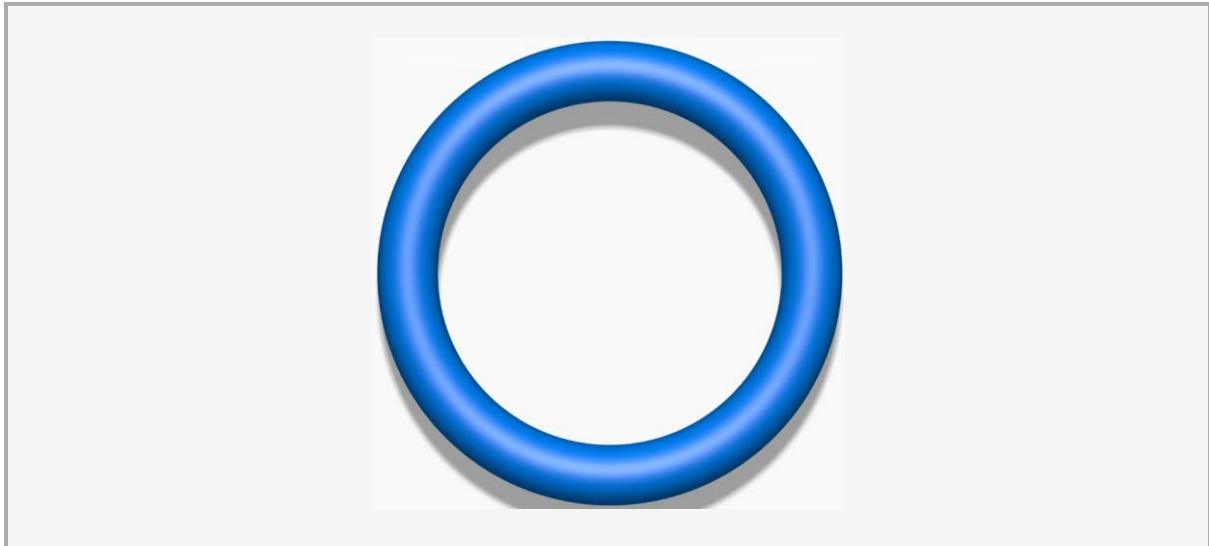


Figure 4: *The unknot is not tricolourable. Since it is one continuous strand, any valid colouring can only use a single colour throughout.*

Since the unknot has no crossings, and the loop is one continuous strand, you cannot randomly change colour in the loop. Therefore, only a single colour is used. It is labelled as not tricolourable.

Next, we can look at the trefoil knot.

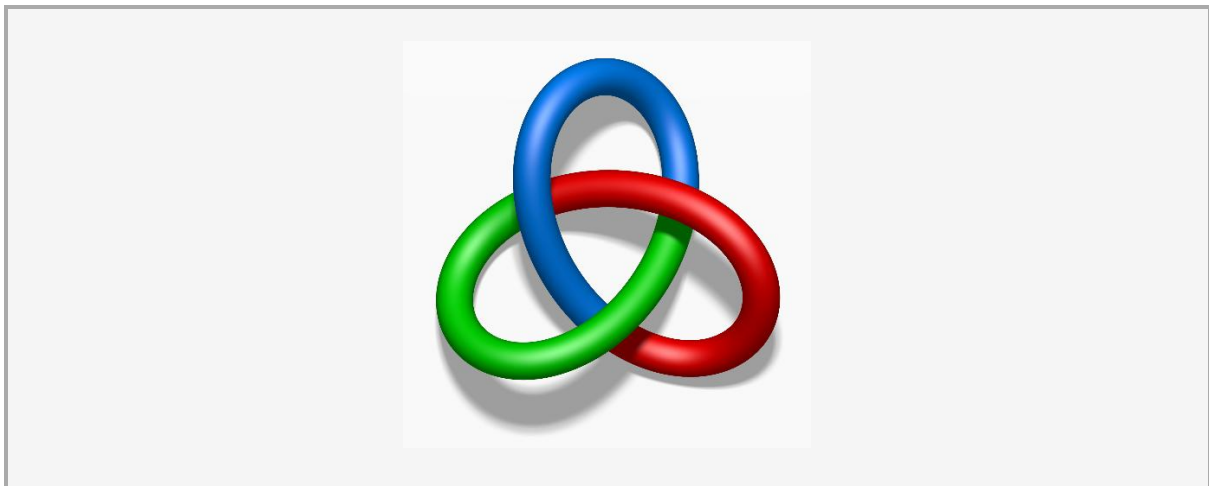


Figure 5: *A valid three-colouring of the trefoil. At each crossing, all three colours meet, satisfying the rule. The trefoil is tricolourable.*

We can colour the three strands of the trefoil red, blue and green. At every crossing, you can see that where the three strands meet, there are three different colours from the three different strands. The rule is satisfied in full. The trefoil is tricolourable.

Even if the trefoil was drawn differently, which then produced a completely different diagram, which made it no longer tricolourable, the fault here lies in the diagram, not the knot itself. The same fault we saw earlier while counting the crossings.

The interesting part is here, involving the preservation of tricolourability before an R move and after. A knot before an R move, which is tricolourable, would also be tricolourable after one of the three R moves, and vice versa.

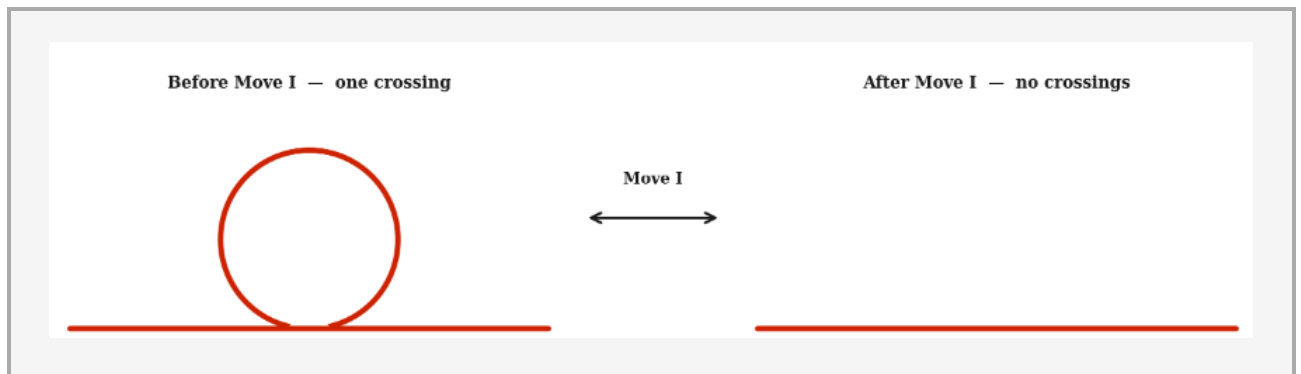


Figure 6: Tricolourability survives R moves. Any legal rearrangement preserves whether or not a valid three-colouring exists.

Therefore, we can say that tricolourability is a genuine invariant – the same knot always has the same tricolourability, regardless of how its drawn and the R moves applied to it.

Now the conclusion snaps into place. The unknot is not tricolourable. The trefoil is tricolourable. Since they have different values of a genuine invariant, they are provably, definitively, irreversibly different knots. This is not a matter of not having tried hard enough. No sequence of R moves, however long, however creative, will ever turn the trefoil into the unknot. It is *mathematically impossible*. That is the power of an invariant. One clever property, computed in seconds, settles what a lifetime of trial-and-error could never resolve.

6. Going Further: The Jones Polynomial

Despite all of tricolourability's greatness, to our shock and horror, there is a better solution...again! Many knots that are distinct from each other happen to share the same tricolourability. Enter Vaughan Jones in 1984, the New Zealander mathematician who discovered greatness: that every knot possesses a polynomial - an algebraic expression in a variable t - that acts as a powerful mathematical fingerprint, often distinguishing between knots. This is called the Jones polynomial, $V(t)$. This can be computed through algebra, and the satisfying solution about this is that different knots produce different polynomials; even though you could have different diagrams of the knot we begin with, we will always arrive at the same polynomial. It was perfect.

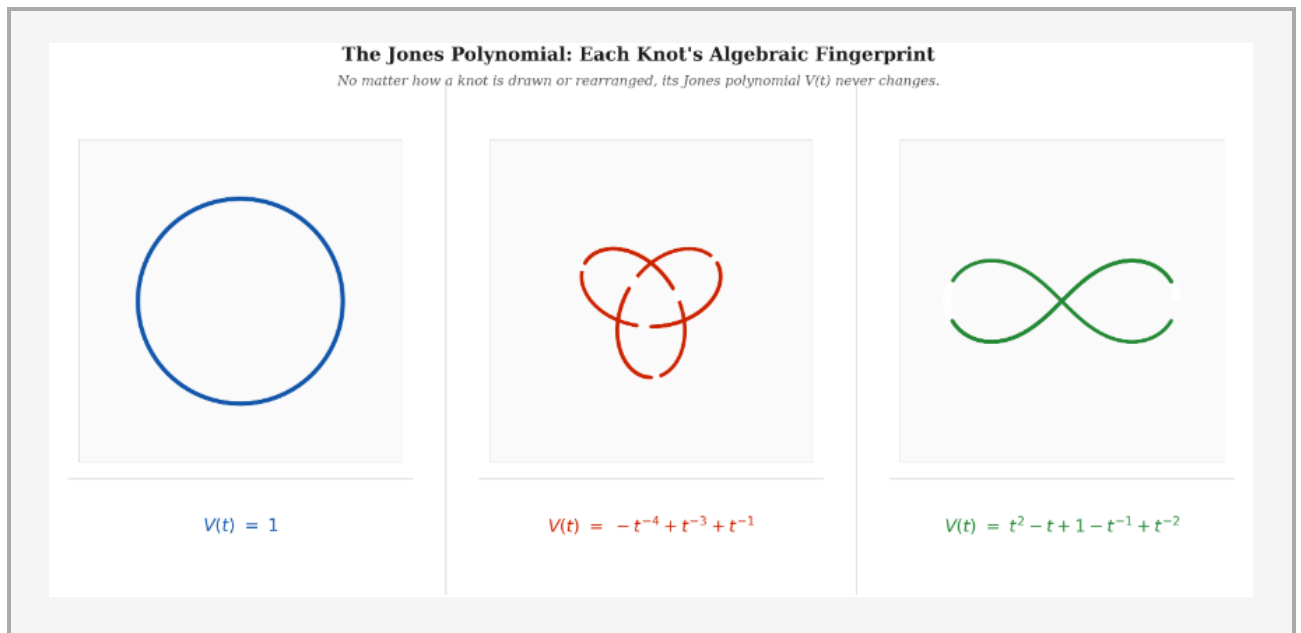


Figure 7: The Jones polynomials of the three simplest knots. Each is distinct — no two knots share the same fingerprint.

To give a flavour of the results, the Jones polynomials of our three favourite knots are:

- Unknot: $V(t) = 1$
- Trefoil: $V(t) = -t^{-4} + t^{-3} + t^{-1}$
- Figure-eight: $V(t) = t^2 - t + 1 - t^{-1} + t^{-2}$

These polynomials are algebraically distinct – the trefoil's polynomial cannot be simplified into the figure-eight's or the unknot's. Jones was awarded the Fields Medal in 1990 for this discovery. The significance was immediate: for the first time, mathematicians had an algebraic tool powerful enough to distinguish knots that had resisted classification for over a century.

7. Why Does Any of This Matter?

While writing this essay, genuinely the thing that impressed me the most is how any of this has any correlation to the real world.

DNA. Inside each of your cells, approximately two metres of DNA is crammed into a nucleus smaller than a pinhead. When a cell needs to read or replicate this DNA, it must unwind the double helix – but the unwinding process creates knots and tangles in the strand. Enzymes called topoisomerases physically unknot the DNA so that it can be read and copied correctly. Cancer researchers are now designing drugs that interfere with topoisomerase function, preventing cancer cells from replicating their DNA, and the mathematical classification of which knots these enzymes create and destroy is a direct application of knot theory.

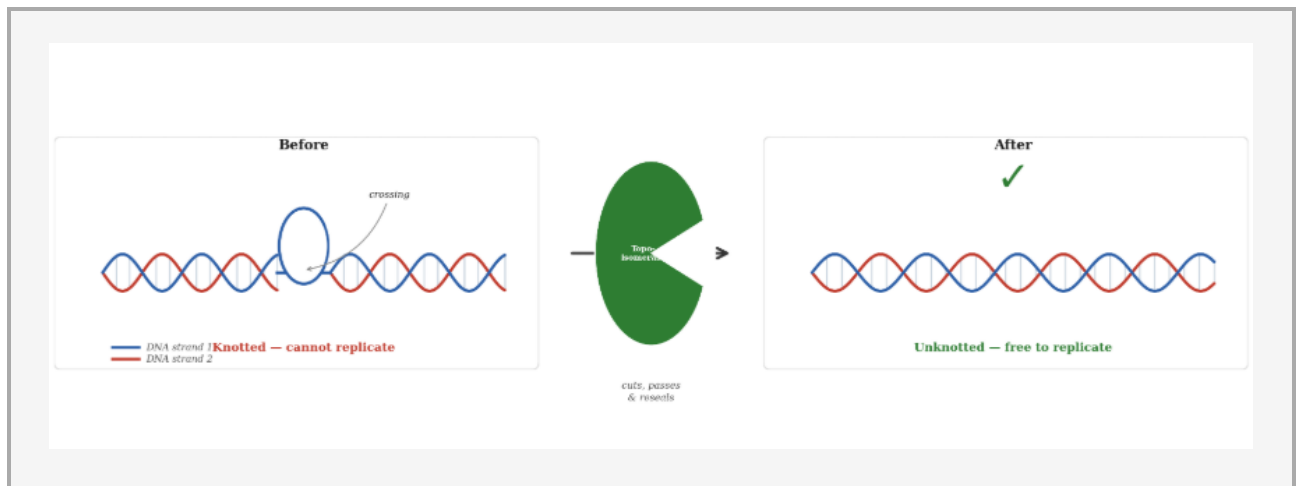


Figure 8: DNA knots form during replication and must be resolved by topoisomerase enzymes. Knot theory provides the mathematical framework for understanding this process.

The role of knot theory is just as prominent in chemistry. In 1989, the first molecular trefoil knot was produced. It was made of actual atoms. Yes, somehow the string-looking thing next to you eventually developed into a whole, different, molecule. The molecule even behaves differently to its unknotted counterpart; and on this basis, chemists are producing machinery around this. What I cannot fathom is how a simple question developed into such scientific complexity.

8. Conclusion

Look again at the simple knot you see on your headphones. This isn't just an inconvenience for people, no. It is the basis for modern oncology and chemistry. The fact that trying to prove that two knots were different would unravel this kind of theory is truly extravagant. But the main takeaway from this essay is not to judge a book by its cover. We shouldn't look at random problems and label it 'easy'. Rather we should do intense algebra on it and then prove its content for ourselves. But truly, the algebraic aspect of this whole problem is so unexpected yet so elegant. The trefoil cannot physically be untangled into an unknot, as we have so many times tried and failed. But this magical force known as algebra, defined tricolourabilities, defined unique polynomials, and suddenly we have a world changing aspect of mathematics.

The best mathematical questions, it turns out, are the ones that look like children's puzzles.

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