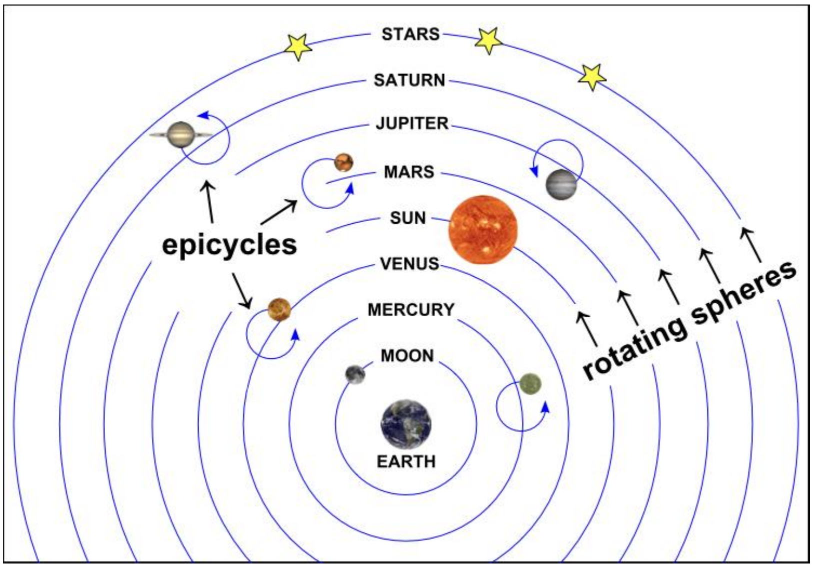
What Goes Around Comes Around, But Not So Round

By Aidan Richardson

Things come in all shapes and sizes, the universe is no exception, constantly expanding like our knowledge about it. For example, how we can predict the sun will expand massively in 5 billion years, and truly become a big, gargantuan, ball of flame, in a phase known as the 'red giant'. Real creative name, I know. But sadly, we probably won't be around to see this as we will most likely be extinct, or if we happen to be around, our huge red friend will eventually engulf us.

It is great how we know all this information and can make predictions about the future. However, there is still a lot we do not know, and at many points we thought we knew a lot but really, we didn't. For starters, in the ancient and medieval times, the Greek Philosophers Eudoxus and Aristotle theorised that our solar system had a stationary Earth at the centre; that had planets orbiting us in perfect circles called epicycles and this theory was the 'Geocentric Model'.



Geocentric Model

They produced this theory with no physical evidence, no mathematical reasoning, just pure idealism, and philosophy perfection.

In around 150 AD, Claudius Ptolemy completed ‘The Almagest’ which brought together all the information thought to be true about the universe, over the last few centuries and that sadly reinforced the notorious ‘Geocentric Model’.

In the 16th century, an astronomer finally came to their senses and suggested that Earth is not at the centre of the solar system. This man went by the name of Nicolaus Copernicus and called this theory the ‘Heliocentric Model’ which he developed through analysing Claudius Ptolemy’s mathematics and seeing many inaccuracies. The reason I say developed is because it was originally theorised by the Greek astronomer Aristarchus of Samos, who calculated the sizes of the Sun and Earth and depicted that the Sun was larger than the Earth. Even though his calculations were quite inaccurate, his idea he was not wrong. Due to the sun being larger than earth, he said that it had to be at the centre of the Solar System. Copernicus did do wonders by reviving the theory and having some sort of evidence to do so. Although, it was still though that the planets orbited in perfect circles.

Heliocentric Model

Finally, in the 17th century, Johannes Kepler produced his 3 Laws of planetary motion which had been enabled by Tycho Brahe's astronomical data of Mars, which happens to be the most precise and accurate data made by observations from the naked eye. This was groundbreaking for the time, as it contradicts the perfect circular orbit everyone had believed for so long and made a huge step in utterly understanding our universe.

Kepler's Laws of Planetary Motion:

1. The orbit of a planet is an ellipse with the Sun at one of its two foci.
2. A line segment joining the planet and the Sun sweeps equal area in equal time intervals.
3. The square of the orbital period of a planet is directly proportional to the cube of the semi major axis of its orbit (The further the planet is from the Sun the longer the time it takes to complete one orbit).

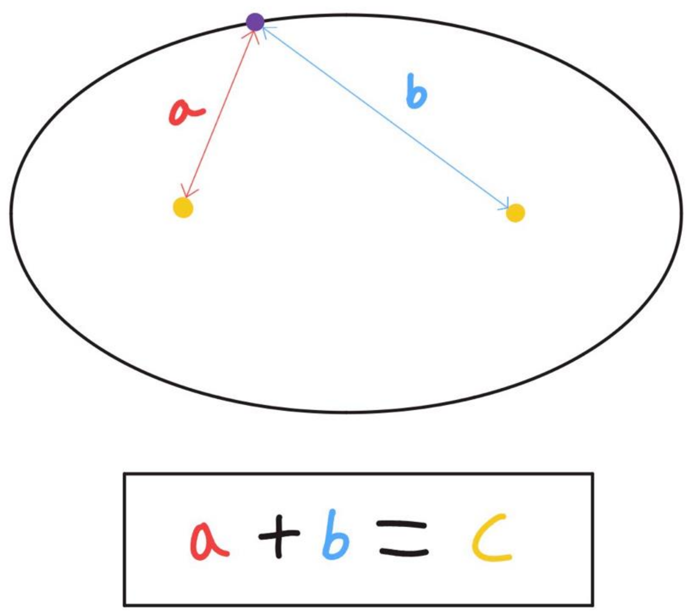
He published this work in his book 'Astronomia Nova' in 1609. Later in 1687 Isaac Newton proved Kepler's First law through rigorous calculus with the use of Newton's own 'Law of Universal Gravitation' and to this day people still prove it through calculus but also through the manipulations of polar coordinates.

However, this is quite complex and there is a much more graphic proof, done by Richard Feynman which utilises geometry and is executed in a way it could be understood by someone with no prior knowledge of complex maths and calculus. It was demonstrated in his guest lecture 'The motion of planets around the sun' (13/3/1964) which happened to be an unpublished partial transcript. Therefore, it did not make it to his free published collection. Instead, it was published later on as a book called 'Feynman's Lost Lecture' by Judith Goodstein and David Goodstein, after they found the remaining pieces of the transcript in Feynman's colleague's office.

Before we dive into the truly fascinating geometric proof, I need to make sure you are up to scratch with what an ellipse actually is…

By its definition 'a plane curve surrounding two focus points, where any point on the plane curve has sum distance from both focus points that is equal to a constant'.

This can be slightly confusing but in simpler words an ellipse is a stretched-out circle or even a squashed circle for that matter, and instead of the focus being on the centre, its focus is on two points called foci. Which is pretty ironic right? Well, the definition does some justice to what the foci are useful for, but a diagram really makes it 60,000 times easier, no seriously, it does as the brain processes visual information 60,000 times quicker than text.



In the wonderful diagram, I drew myself, lying right above is an ellipse with its foci points in yellow (foci meaning plural for focus in Latin) and a point on the ellipse's curve in purple. Then 'a' is the distance from the first focus point to a point on the ellipse curve and 'b' is the distance from the second focus point to the same point on the ellipse curve.

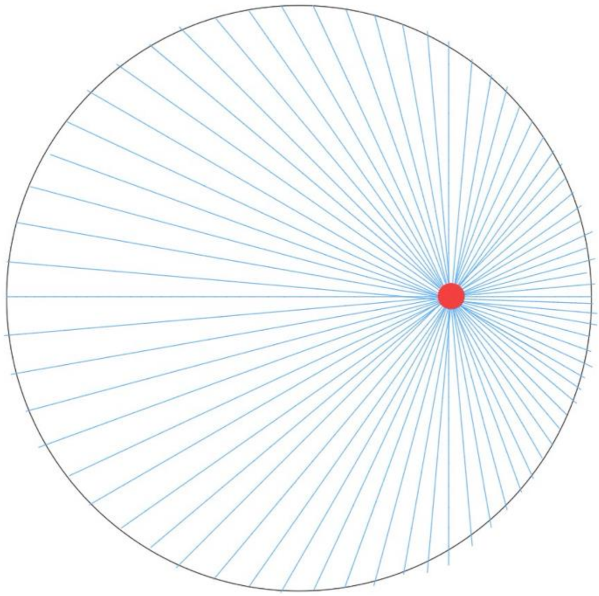
Once we have that clarified, for the shape to be a true ellipse, no matter where the purple point is on that curve, the sum of 'a' and 'b' (a + b) will always equal the same number which us mathematicians like to call a constant, that's why, not coincidentally in the diagram I have labelled the sum of 'a' and 'b' as 'c'.

Since we all up to speed with what an ellipse should be…

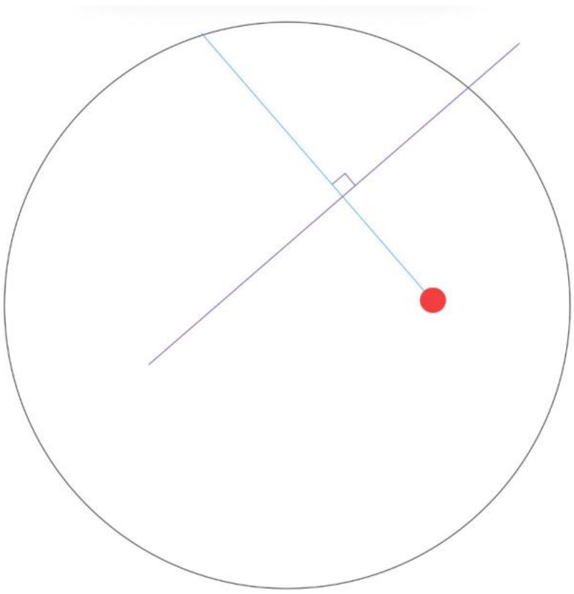
THIS IS WHERE THE FUN BEGINS!

Just before we start, I will mention some of this may be quite challenging to wrap your head around but should not require much external knowledge, so stay along for the ride, to learn what shape our Earth's orbit around the Sun really is, since we are very much attracted to it (well about 9.81ms^-2).

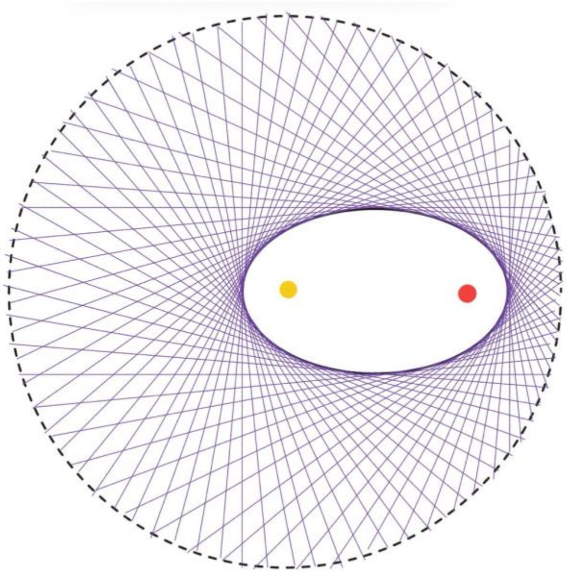
Let's start with a circle and plot a point somewhere in our circle that is not in the centre, we will call this the eccentric point. Where we will connect many lines from this point to the circumference giving something like this:



If we were to take a line and rotate by the middle point of the line, between the circumference and the eccentric point it would look like this:

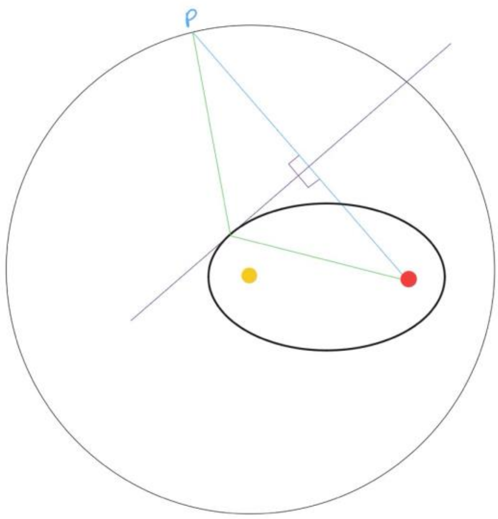


You may know these are what we like to call perpendicular bisectors, so now let's apply this to all our lines:

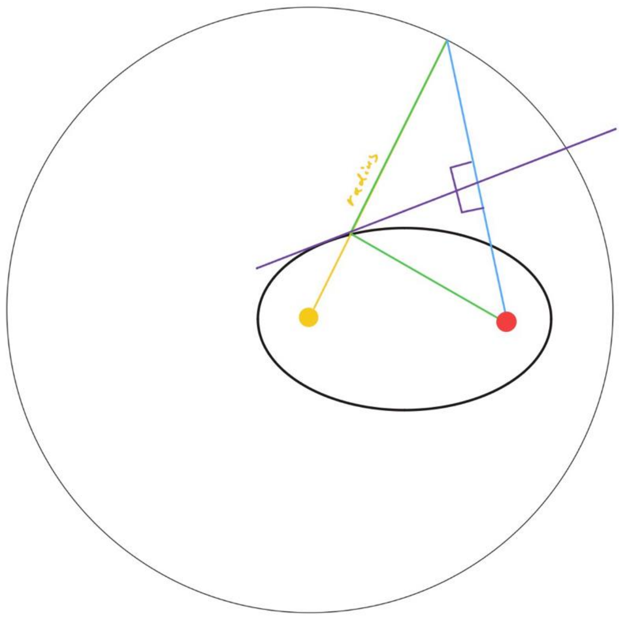


You should be able to notice this forms a perfect ellipse where our perpendicular bisectors are tangent to the ellipse, with our eccentric point as one focus point and the centre of the original circle as the other focus point. Another fun fact the word focus comes from the Latin word foco which means fireplace, due to in our orbit one of the focus points is the Sun and it is viewed as the fireplace of our solar system. But how can we truly tell this is a perfect ellipse.

Narrowing our view on to one of these lines and its perpendicular bisectors again, we label P the point (where our initial line from eccentric point to the circumference of the circle meet) and drawing two lines in green one that joins any point on the curve of the ellipse to point P and the other from the same point on the curve to our eccentric point:



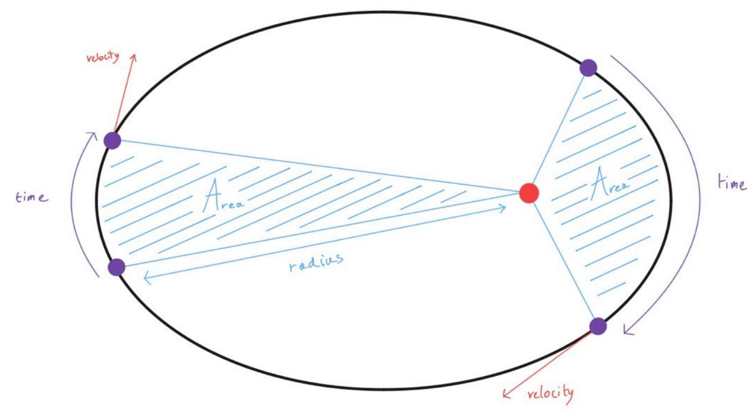
Carefully, looking at these, they form two congruent triangles. Now we need to include our centre point by utilising its radius, we will do this by rotating our diagram so that our line from the curve of the ellipse to point P follows along the radii:



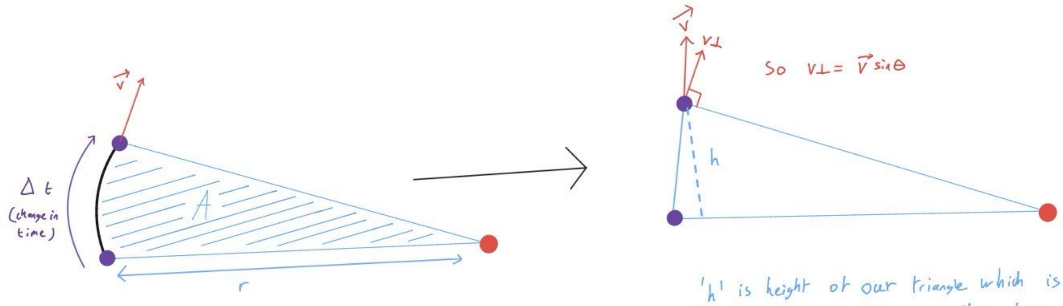
The line from the eccentric point to the curve of the ellipse in green, intersects the radius which happens to be at a point on the curve of the ellipse. This should look familiar to the diagram of a perfect ellipse we drew earlier where lines ‘a’ and ‘b’ (in this case the green and yellow lines) sum to give a constant, this is exactly the same.

We are now going to resurface Kepler’s 2^{nd} Law:

Kepler's 2^{nd} Law: A line segment joining the planet and the sun sweeps equal Area in equal time intervals.



Focusing on these areas at very small-time intervals, since the arc of the curve will be remarkably close to a straight line, we are going to assume its shape makes a triangle:



$\vec{v}$ is the actual velocity the planet is travelling at.

$v\perp$ is the velocity it travelling perpendicular to the radius.

'h' is height of our triangle which is the displacement between the two radi.

$\Rightarrow h = v\sin\theta\,\Delta t = v\perp\,\Delta t$

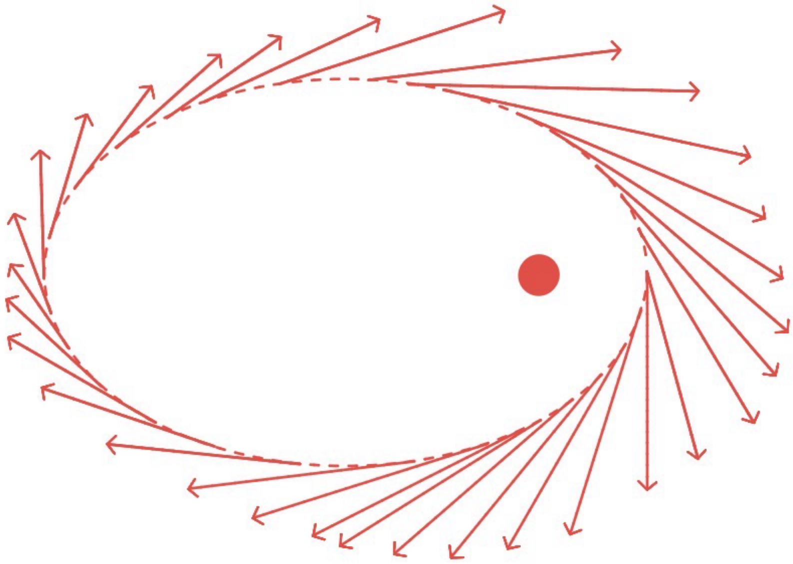
$$\text{Area} = \frac{1}{2} \times \text{base} \times \text{height}$$

$$\Rightarrow \text{Area} = \frac{1}{2} r v\perp \Delta t$$

Since $rv\perp$ is equal to Angular moment and gravity produces no torque, we can say Angular moment is constant.

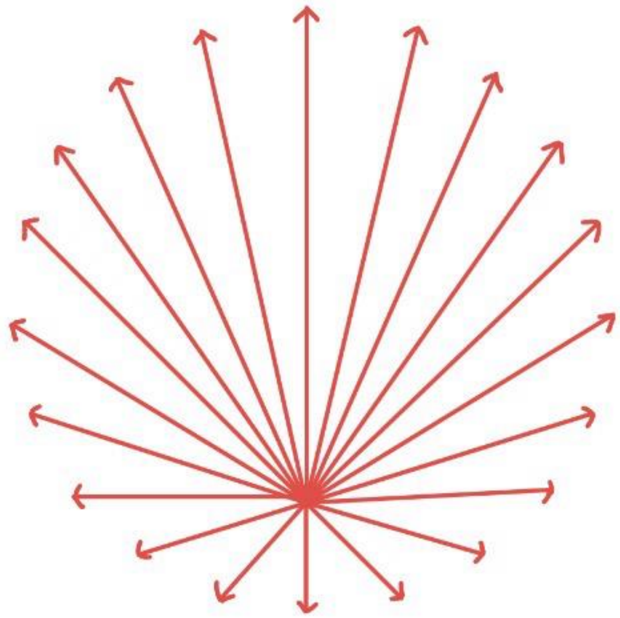
$\therefore$ This then means Area $\propto$ Time

Extracting the velocity vectors of the planet telling us its direction and speed at points of its orbit around the sun:

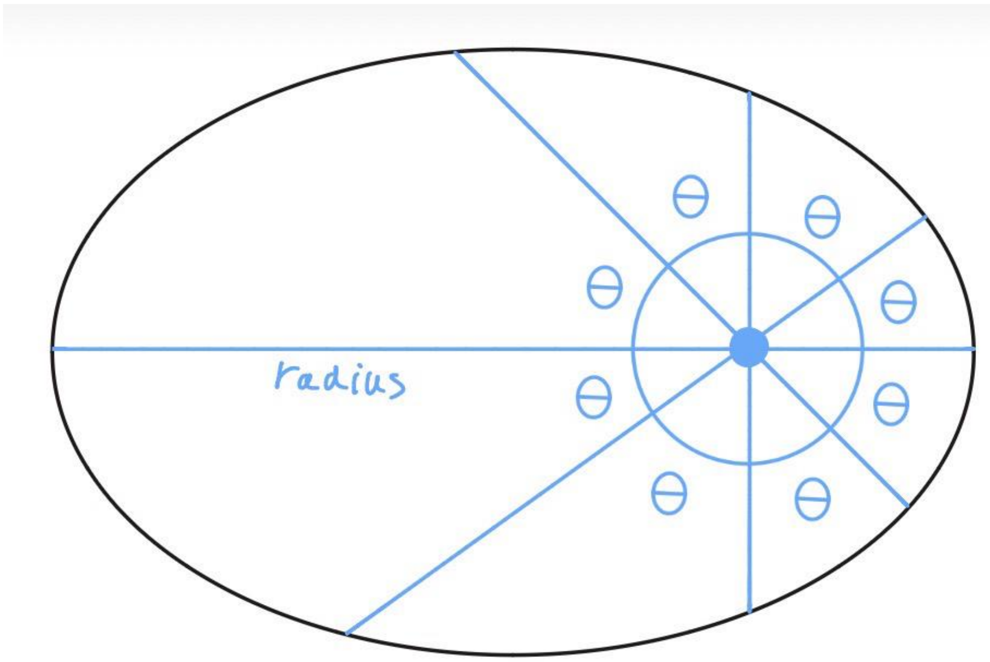


It shows us that the closer we are to the sun (which is at the eccentric point) the faster the planet travels while the direction is changing constantly but staying tangent to the curve.

Now if we take these velocity vectors and make them meet the end of their tails at one point it forms a perfect circle:



Taking its suggested path of orbit and splitting it up into equal slices of angle θ degrees (these slices would be smaller but to make it easier to visualise we will use bigger ones):



Implementing this diagram alongside the use of the information we stated about Kepler's 2nd Law and now Newton's Law of Universal Gravitation:

due to $\text{Area} = (\text{radius})^2 \pi$ (for a circle)

we can say $\text{Area} \propto (\text{radius})^2$

Since $\text{Area} \propto \text{Time}$

$\Rightarrow \text{Time} \propto (\text{radius})^2$

Newton's Law of Universal Gravitation: $F = G\frac{m_1 m_2}{r^2}$

$\Rightarrow \text{Force} \propto \frac{1}{(\text{radius})^2}$

$\text{Force} = \text{mass} \times \text{acceleration}$

$\Rightarrow \text{Force} \propto \text{acceleration}$

$\Rightarrow \text{acceleration} \propto \frac{1}{(\text{radius})^2}$

$\text{acceleration} = \frac{\Delta\ \text{velocity}}{\Delta\ \text{Time}}$

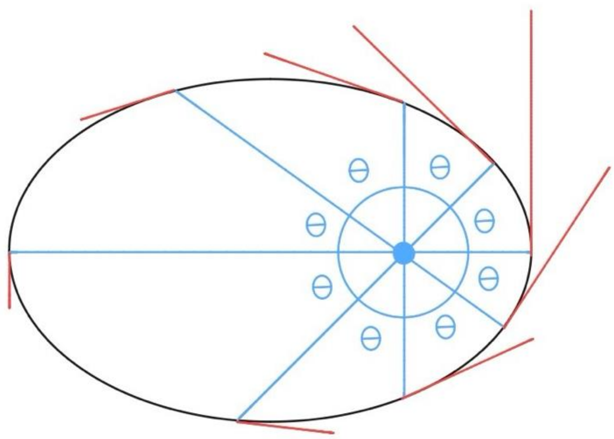
$\Rightarrow \frac{\Delta\ \text{velocity}}{\Delta\ \text{Time}} \propto \frac{1}{(\text{radius})^2}$

$\Rightarrow \Delta\ \text{velocity} \propto \frac{\Delta\ \text{Time}}{(\text{radius})^2}$

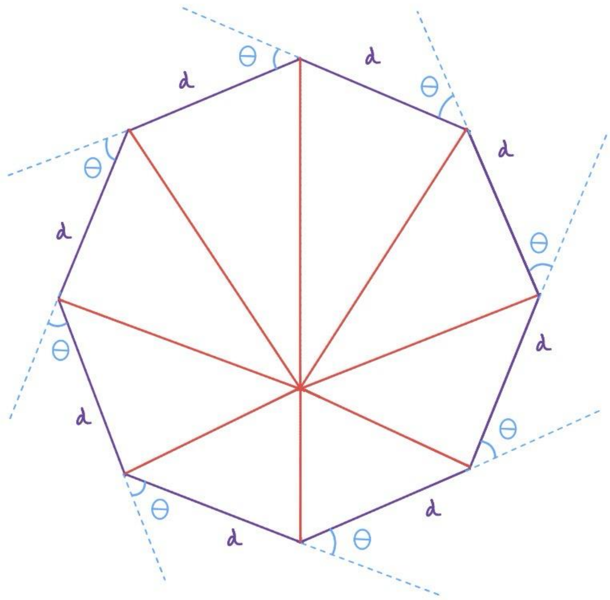
$\Rightarrow \Delta\ \text{velocity} \propto \frac{(\text{radius})^2}{(\text{radius})^2}$

This tells us that velocity is directly proportional to some constant.

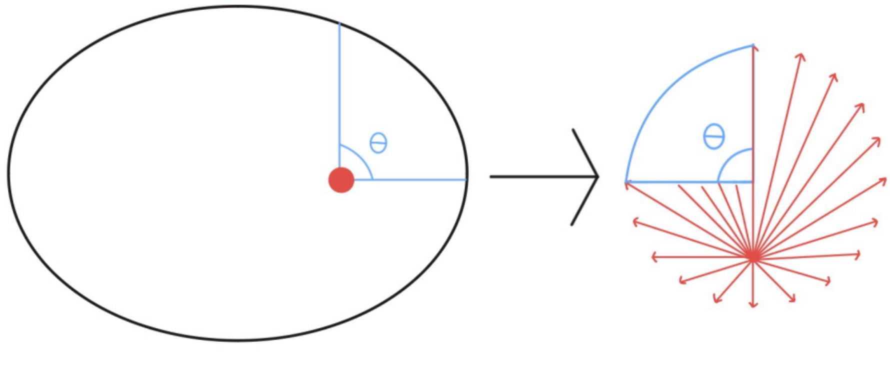
So, we need to investigate what this constant could be, but we do know its directly proportional to velocity, so we will look there first. Going back the last diagram, but this time including the magnitude of its velocity at the beginning and end of each slice:



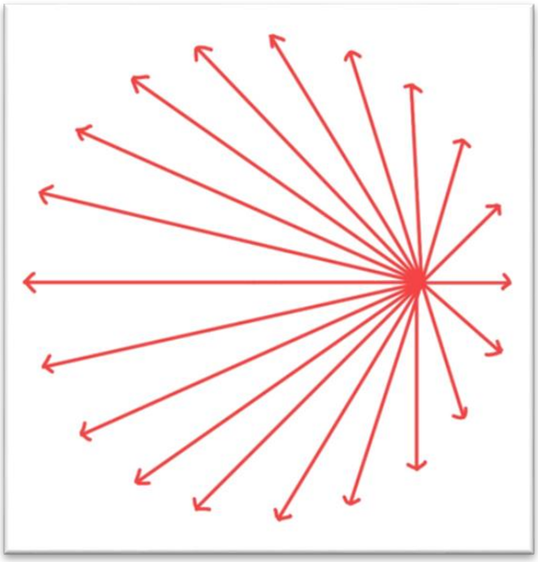
Then repeating the exercise of meeting these velocity vectors' tails at one point, and looking at the distance between each tip, we can see that the distance is equal for all of them. Which by connecting each tip creates a regular polygon which means the external angles of the polygon must all add up to 360 degrees, so this means each external angle is θ degrees. Showed here:



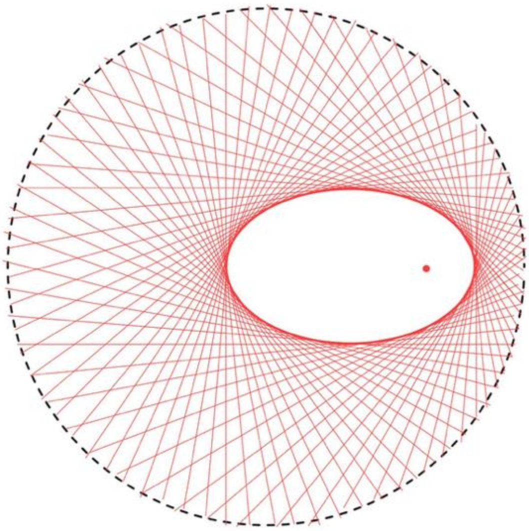
So, if we were to travel around the orbit by θ degrees from the horizontal in respects to the sun, it correlates to travelling θ degrees around the velocity vector circle we created earlier from the top (since this is where the vector lies that correlates to the horizontal point in respects to the sun). Where the slice with angle θ degrees stops on the orbit curve, is where the vector touching θ (in yellow) corresponds to:



Now taking the velocity vector circle again and rotating it 90 degrees anticlockwise:

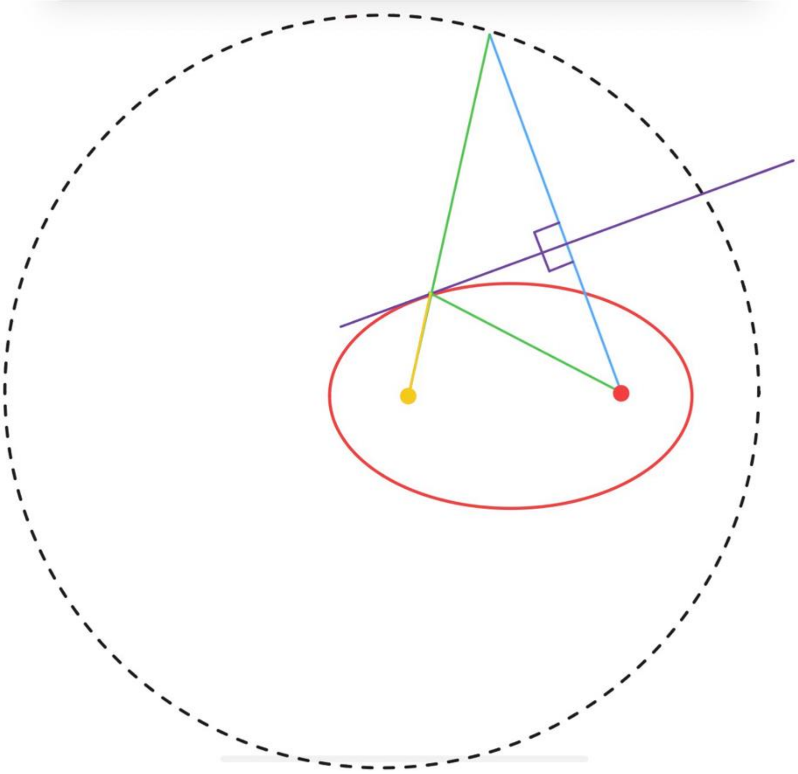


Then taking all the perpendicular bisectors to all the velocity vectors:



Are you getting Déjà vu? I sure am. It looks exactly like what we had at the beginning, a perfect ellipse.

Using the same geometry proof as before:



So finally, since we have two lines one from the centre point and the eccentric point meeting at a common point on the circle's circumference. The perpendicular bisector to the eccentric line is tangent to the ellipse. Then the point of tangency is where the perpendicular bisector intersects the radius from the centre point to the common point.

This means that the point of one ellipse which is θ degrees from the horizontal in respect to the centre of the circle has a tangent shape perpendicular to the eccentric line. Therefore, the perpendicular bisector is parallel to the velocity vector. So, this emergent curve inside the velocity vector circle has the exact tangency property we need our orbit to have.

Hence the shape of the orbit must be an ellipse.

Here is the thing the eccentricity of the orbit (how stretched it is from 0-1) is about 0.0167, while a circle’s is 0. So, they were not far off when the predicted orbits were in perfect circles.