What is a Fixed Point?: Explanation and Applications

-

Leo Knowles

-

TRM Essay Competition 2026

Contents

1. The Topological Perspective

The typical person hearing the statement ‘a donut is the same as a coffee mug but not a dining plate’ would understandably be confused. For starters, eating your mug may be a difficult process but even attempting to think past the usual backwards logic of the statement would result in the argument that a plate and a mug (both being kitchen based items) are more similar. However, this is not considering the function or appearance but instead the shape. The statement from a topological perspective makes sense. That's because Topology is the study of properties of geometric shapes that remain the same under continuous deformation without tearing or gluing. Put simply: the relaxed geometry of extremely squishy shapes.

Imagine compressing the part of the mug where you'd drink out of and then stretching the handle into a donut. Topology uses the fact that even though they appear different, their spaces are identical so they can be considered as the same topological space.

This raises questions on the way in which we define whether or not different ‘objects’ can be deformed into each other without adding or removing substance. If two shapes do satisfy this they are called homeomorphic. A common value of Euler's Characteristic (χ) decides this. This is when V is the number of vertices in a shape, E is the number of edges and P is the polygonal faces and:

$$\chi = V_s - E_s + P_s$$

Considering the previous example has no easy way to measure edges, vertices or faces we need to use triangulation. This is the process of practically turning a round surface spikey so it can be defined with a Euler Characteristic. Take a cube and a tetrahedron. Their characteristics can be found by:

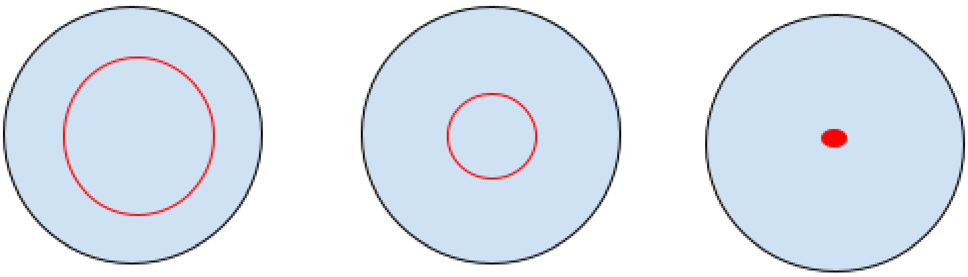
Cube: $\chi = 8 - 12 + 6 = 2$

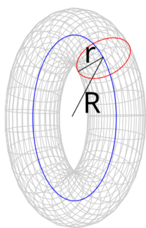
Tetrahedron: $\chi = 4 - 6 + 4 = 2$

As these shapes both have the same value (2) for their Euler Characteristic we know they share the same topological properties such as: connectedness, compactness, separation axioms. Extending this for more complex shapes means that they can be studied by comparison to a more simple one.

Another way in which topologists categorise things is by the number of distinct loops you can draw around the surface of a shape. This collection of loops is called the fundamental groups of a shape.

Any loops drawn on the surface of a sphere can be contracted into a singular point meaning that all loops on a sphere are technically the same. It can be concluded that a surface loop on a disk (circle) can be therefore considered 'trivial' or '0' as it has no holes once contracted.





In contrast, loops going around the shape of a donut could have many more (as one could go around 2 and one could go around 3 times, meaning the two could never contract to the same shape). This is indicated by the red (r) on the diagram to the right. Therefore, it can be said the fundamental group is $\mathbb{Z}$ (some integer).

2. The Fixed Point

A fixed point in one that maps to itself under a function. For example:

$$sin(x) = x,\ x = 0$$
$$sin(0) = 0$$

Fixed points can be used in iteration to slowly close in on a solution or particular point, in applied fields like economics or mechanics using the upcoming fixed point theorem or in fractal geometry such as the Mandelbrot set.

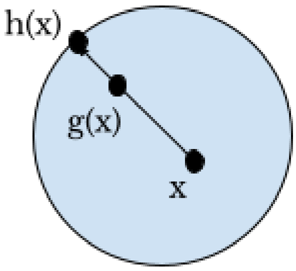
3. Brouwer's Fixed Point Theorem

Brouwer's fixed point theorem states that for any continuous function mapping there is at least one fixed point, a point that remains in its original position after the mapping. This can be proved by contradiction:

Let's assume that Brouwer's Theorem is not true meaning there is an existing transformation that means no point ends up at the same place as it began. Or:

$$x \neq g(x)$$

In a disk, by drawing a ray (arrow) from the start point to the end point we create a new function (h(x)), which assigns every point in said disk to the boundary, or outer region of that disk.

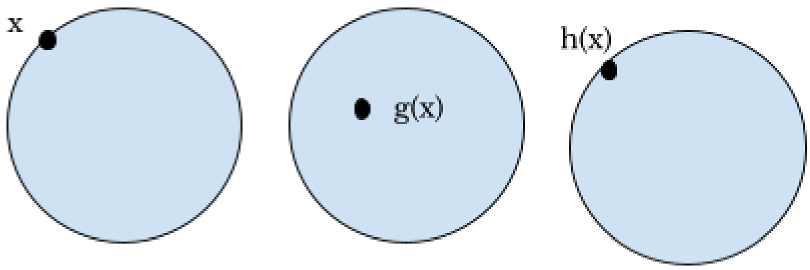


This diagram shows the starting point of g(x) to x which is then mapped to the boundary by h(x). However, since we have assumed that $x \neq g(x)$ then if x is on the boundary to begin with g(x) must be inside the disk. Drawing a ray through these points will subsequently mean that the boundary is met for the first time from the starting point and therefore is met during the transformation h(x).

This must follow the pattern of:

$$x \Rightarrow g(x) \Rightarrow h(x)$$

But if on boundary $x \Rightarrow g(x) \Rightarrow x$



Earlier we defined that the disk can be viewed algebraically so trivial or 0 and the boundary (like the aforementioned torus) can be assigned any integer ($\mathbb{Z}$). But this therefore means that any integer is mapped to 0 and then is mapped to another other possible integer. Overall, this shows there is a composition in which each integer is not assigned to itself however, topologically we discovered h(x) = x. This therefore is a contradiction and we can conclude that the theorem is true.

This is used whenever you stir a coffee, regulate your body temperature or shuffle a deck of cards. Brouwer proved that it's not by coincidence or luck that at least one card will be in the exact same position or a molecule (or point) in the coffee will remain in the exact same place.

It’s commonly questioned that if you stir the coffee, then stop stirring then begin to stir again. Then the point would have changed. However, this is not considered one complete continuous transformation so would not hold.

It also means scrunching a piece of paper and placing it over an scrunched identical piece will lead to two points being exactly above each other (which can be tested yourself).

4. Usage in Nash Equilibrium

John Nash utilised this fixed point theorem to create: Nash Equilibrium. This is the state in game theory in which neither of two players can benefit by changing their strategy, assuming the other players stays constant. A common example is the prisoner dilemma:

X	Suspect A		
Suspect B	X	Stay Silent	Betray
	Stay Silent	1 year, 1 year	10 years, 0 years
	Betray	0 years, 10 years	5 years, 5 years

This can be written as a pay off matrix:

$$[((-1,-1)(0,-10)(-10,0)(-5,-5))]$$

The Nash equilibrium is the one highlighted in orange as both suspects are at a point where they can stably benefit. No suspect wants to individually change their choice.

This was proved by a special case of the fixed point theorem: Kakutani's fixed point theorem. The difference is that this considers a set of points meaning that an input is inside its own output set:

$$x \text{ is in } \varphi(x)$$

Nash equilibrium is beneficial in modern economics because it removes the problem of firms and markets considering what each other will do in an infinite loop. In any market, each firm's optimal price depends on its rivals' prices and each trader's optimal bid depends on others' bids. This mutual dependence seems to create an infinite cycle with no decision being made. Nash creates an outcome that takes any combination of strategies and returns the set of optimal replies for every player simultaneously. The existence of such a point where this happens is found by Kakutani's fixed point theorem. Kakutani guarantees at least one fixed point must exist. This matters enormously for economics because it means that in any finite game like any market, auction, or trade there is always at least one prediction that does not contradict itself. Without the fixed point guarantee, Nash equilibrium would be a useful concept that might apply nowhere; with it, economists have a universal foundation for predicting behaviour across financial markets, international trade negotiations, and other parts. But it all comes from the same topological argument that a continuous map on a compact space cannot avoid having a point it sends back to itself.

5. Iterating Fixed Points

A common numerical method used to find the root of a function is called fixed point iteration. This takes an approximate root and repeatedly plugs it into a function to find a fixed point.

$$x_0 \rightarrow x_1 = g(x_0) \rightarrow x_2 = g(x_1) \rightarrow x_3 = g(x_2)$$

There are two outcomes for this process:

$$|g'(x)| < 1$$

The curve is shallower than the diagonal at the crossing meaning nearby points get squeezed closer on each step. The fixed point is attracting and iteration converges.

$$|g'(x)| > 1$$

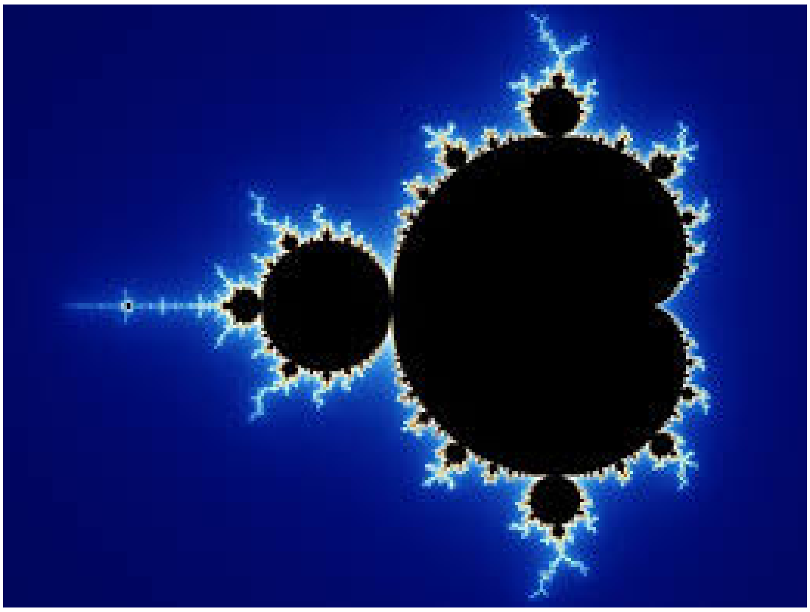
The curve is steeper, nearby points get stretched further away. The fixed point is repelling and iteration diverges meaning it wont come to a single solution and in the case wont work.

6. Fractal Geometry Of Fixed Points

By using this iteration of fixed points, beautiful images can be made called fractals. Most notably the Mandelbrot set, in the complex plane, which is defined as:

$$z_{n+1} = z_n^2 + c$$

For some values of c, the sequence stays bounded and converges but for some others the terms go to infinity. By colouring all values of c that stay bounded black and apply a gradient to the others the shape below is reaped.



There are many different other sets that use fixed points and iteration such as the Julia Set or Newton Set. They all exist to depict how simple actions can cause such strange and surreal images.

7. Conclusion

Overall, this essay showed how something so simple such as the idea of a singular point or set of them in a function can create such beauty in mathematics or in real world applications like economics or engineering. It expresses how even simple things like stirring coffee or scrunching paper can be influenced and discussed mathematically even if not thought about that way.