## Introduction

What is a shape? This question seemed simple at first when I thought of it. I mean, everyone knows what a shape is, right? However, as I thought more and more about this question, I came to the realisation that there isn't really any simple, intuitive answer. This became more apparent as I turned to the internet for help, learning that there is a lot more to the question than I first thought. I hope that, after reading this essay, you will gain a deeper appreciation underlying something as seemingly simple as a shape.

## What a shape is

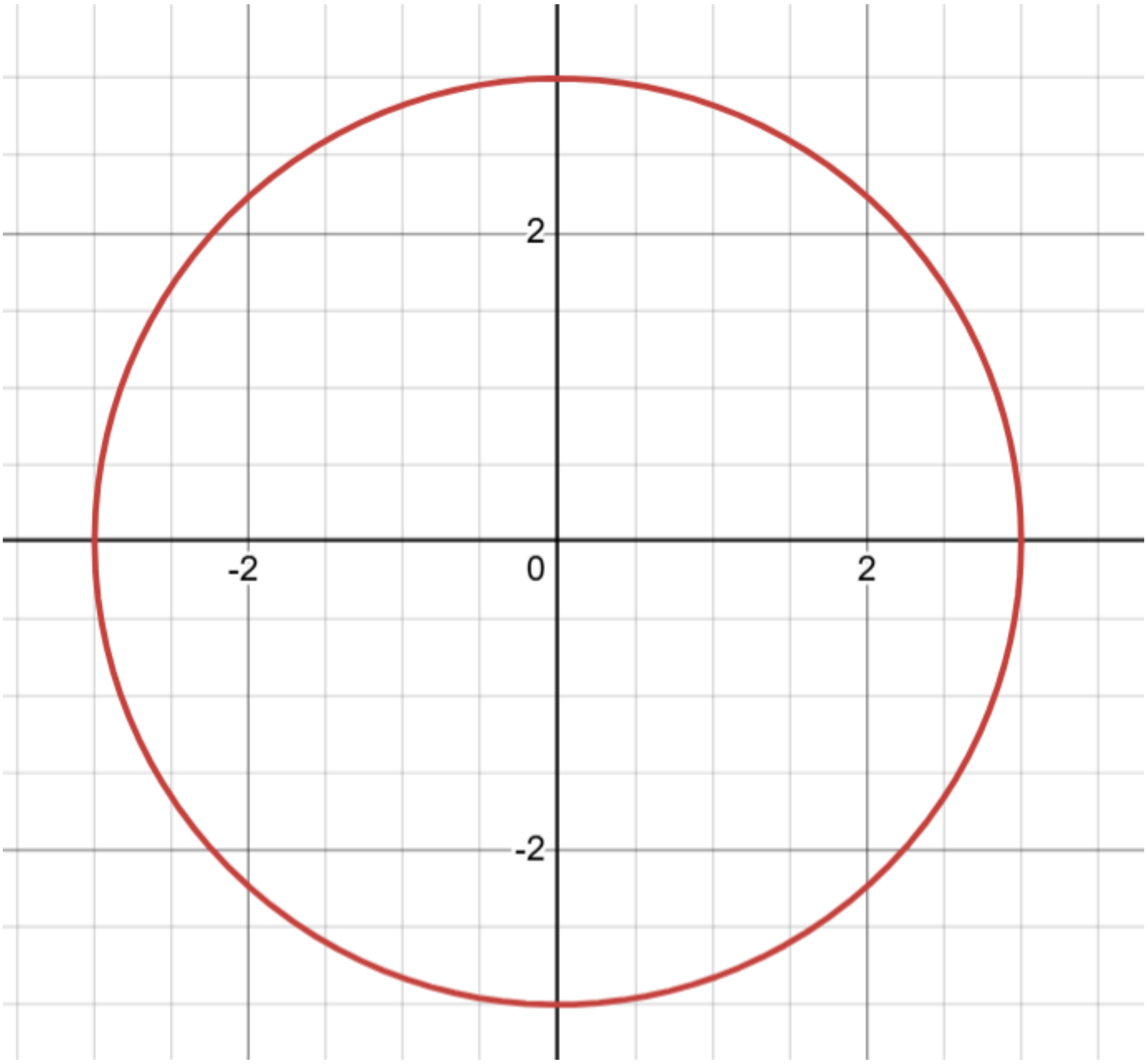
I will begin by discussing the definitions that other people gave me with varying qualifications. I asked each person to give me their best answer to the question "What is a shape?" First, the A-Level Maths students' understanding of a shape: "a shape is a physical body configured of line(s)."

It made sense that the Maths student would turn to a geometrical idea of a shape, however this definition is not entirely correct. To begin, a shape does not have to be a "physical body" ; it can be entirely conceptual, and thought about analytically. The idea of "lines" also seemed too limiting here. Given the amount of circle geometry in A-Level Maths, it was surprising to see that the student didn't consider this (the problem being that a circle is not made of lines, rather arcs). With this being said, given the level of an A-Level Maths student, it is not a bad definition, and close to what I expected.

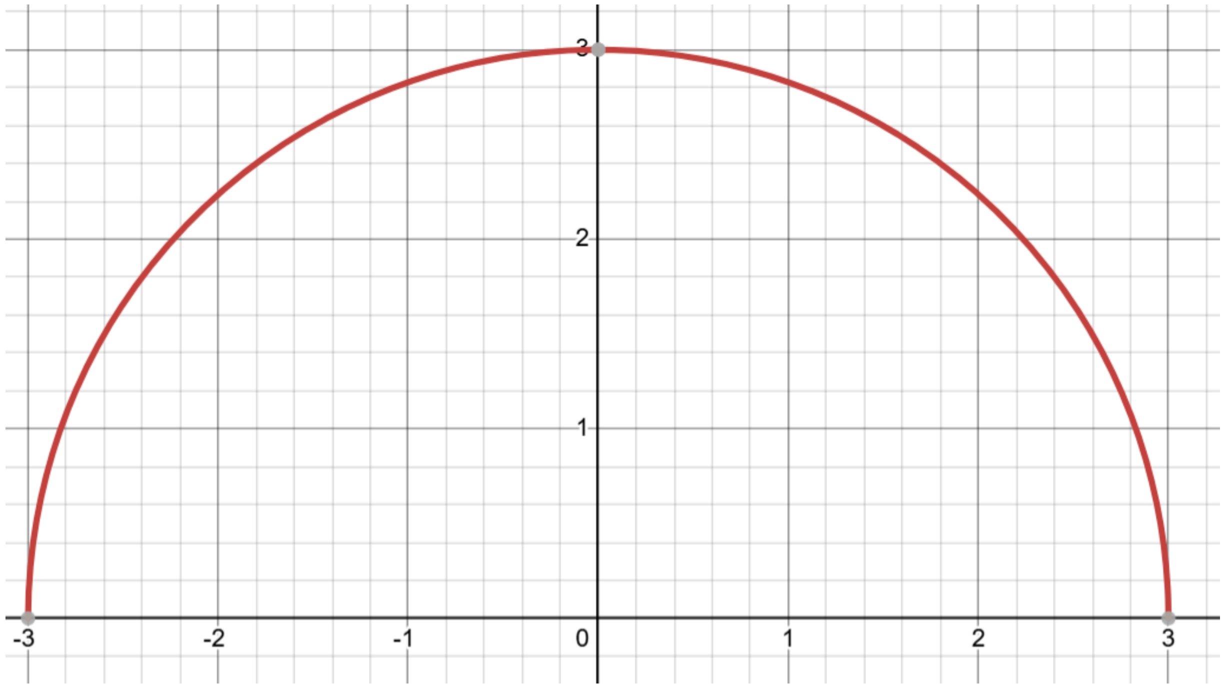
Next, we move onto the Maths teachers' definition of a shape: "an area enclosed by edges." This definition also has its issues. Consider an object like a bowl, it is not "enclosed", as it has an open "side" [1], yet it can still be thought of as a shape. This may be nitpicky, but the choice of the word "area" is also not entirely accurate. A line or a dot can be thought of as a shape, however it does not conceptually have to have an area. This brings me to the final, and by far, best definition of a shape.

The Physics teachers' definition of a shape: "a shape is a closed surface in an $\mathbb{R}^n$ space". Let's break this definition down. A "closed surface" is quite self explanatory. Essentially, this definition considers the following as a shape:

[1] I would like to note that "side" is probably not the best choice of word here. It was simply chosen for clarity.




And the following as not a shape:




The $\mathbb{R}^n$ space is best explained by examples. The $\mathbb{R}^2$ (as seen above) space refers to the x,y coordinate system, i.e. a 2D space. Any 2D shapes can fit in here, like a square. The $\mathbb{R}^3$ space refers to the x,y,z coordinate system, i.e. a 3D space, subsequently any 3D shape can fit in here, like a sphere, and so on as n increases.

There are two problems with the Physics teacher's definition presenting the idea of a "closed surface". First of all, a shape does not have to be closed (refer to the bowl example discussed previously) and a shape does not have to have a "surface", for example, it could be a point or a line. This definition also restricts a shape to the euclidean geometry[2] ($\mathbb{R}^n$) space, whereas in Maths, a shape can often be seen as more abstract.[3]To put it simply, this definition is too restrictive of how a shape can be represented for every area of Mathematics.

So, what is the definition of a shape? One relatively simple "correct"[4] definition of a shape is what remains invariant under a chosen set of transformations. What does this really mean? Invariant simply refers to something that is never changing.[5] The "chosen set of transformations" means that mathematicians decide which transformations are relevant depending on the specific context of the problem. As an example, consider a triangle drawn on a sheet of paper.

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[2] This simply refers to the "normal" space where shapes obey ordinary rules of distance and size.
[3] That is, a shape can be discussed through its properties and relationships, not necessarily just something drawn in ordinary space.
[4] This definition is a reasonable Erlangen-inspired way to think about a shape. There is no one universally agreed upon definition.
[5] I'd like to note that this is not the mathematical definition of invariant. Invariant refers to something that stays the same under the transformations we are considering. I have chosen the definition here for simplicity.

You can rotate the paper, slide it across a desk, and you will still have the same shape. However, if we were to stretch the paper, the shape will have a different length and therefore is no longer invariant (it has changed).

**Category Theory**

Category theory is a very advanced area of Mathematics. For the purpose of this essay, we will focus only on the relevant parts for the definition of a shape.

I suppose a sensible place to start is by asking what is category theory? Mathematicians work with a large variety of mathematical constructions, each has properties specific to a particular area of study properties that make each area distinct. Simply put, mathematical constructions are the things mathematicians create or define so they can study them. Some examples of constructions are numbers, shapes, equations. From this, properties arise. Take numbers as an example. Some properties that can arise from this is that when you add two even numbers, the answer is always even, multiplication is commutative. Different areas of mathematics study different constructions. To continue with this example, the area of maths concerning numbers is called number theory. Geometry studies shapes and their properties. Algebra studies equations and patterns.

Often in Mathematics, there are links between these fields. Category theory is a way to formalise this link. For simplicity, I will not go through the mathematical definition of a category. The only relevant part you need to know is that each category has objects and morphisms. In category theory, an object is just the “thing” in a category. Objects represent the mathematical constructions that we want to study. For example, if we are thinking about numbers (to be the construction we want to study) each number can be treated as an object. Morphisms describe a relationship or transformation between two objects. For this essay, the easiest way to think about a morphism is a transformation that is considered to be meaningful in certain mathematical settings.

Now, we can link this to the definition of a shape. If a shape is what remains invariant under a chosen set of transformations, category theory helps us define that “chosen set of transformations” by telling us which morphisms are relevant. Essentially, changing the category changes which morphisms matter, and changing the morphisms changes what counts as the same shape.

**Conclusion**

I do hope you found this essay interesting. Rather than focusing on its length, I tried to introduce some interesting topics, and speak a bit about how such a simple sounding question can prove to be quite complicated. If you would like to learn more about category theory, please refer to Oliver Lugg’s video entitled “A sensible introduction to category theory”. Thank you for reading.