What is the difference between topological dimension and Hausdorff dimension?

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Abstract
This essay examines the limitations of the topological dimension when applied to fractal and irregular structures and explores how the Hausdorff dimension extends classical notions of dimension. By comparing topological, similarity, and Hausdorff dimensions, this essay demonstrates why non-integer dimensions are fundamental for describing the complexity of both natural and artificial structures, such as plants and urban areas.

Introduction
Classical geometry assigns integer dimensions to shapes: lines are one-dimensional; surfaces are two-dimensional and solids are three-dimensional. This rule works for smooth, regular objects; however, it is inadequate for irregular sets that exhibit self-similar structure across scales, such as coastlines, leaves, and mountains. Fractals are self-similar shapes that maintain complexity under magnification, and Benoît B. Mandelbrot showed that they can describe many natural phenomena[1]. Certain fractals are too irregular for integer dimensions, prompting Felix Hausdorff to develop a new method to measure the "roughness" of a fractal - the Hausdorff dimension - which can take non-integer values.[2]

1 The Topological Dimension
"The topological dimension or Lebesgue covering dimension of a topological space is defined as the minimum value of n, such that any open cover has a refinement in which no point is included in more than n+1 elements."[3] Intuitively, this means that the topological dimension measures the amount of independent spaces a direction has, based on how you can cover it with overlapping sets. For example, a line has topological dimension one, a plane has topological dimension 2 and a cube has topological dimension 3[4]. This formal definition provides a way in which we can describe dimensions for regular shapes, but as discussed above, it does not capture the complexity of irregular, fractal objects such as the Koch curve or the Sierpinski triangle. The topological dimension is fundamentally flawed for this reason, take the example of the Koch curve. According to the previously stated definition, it would be assigned a topological dimension of 1, as it is a continuous curve where, locally, at a point, you can only move in one direction. So even though the Koch curve is infinitely intricate, the topological dimension seemingly reduces it to the simplicity of a straight line.[5]

2 The Hausdorff Dimension
Despite its limited use within fractals, the topological dimension is useful for providing simple measures of dimensionality for regular shapes such as planes and cubes. It also serves as the foundation of more advanced concepts - fractals, which defy the intuitions of classical geometry.[6] However, for this reason, the Hausdorff

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[1] Benoît B. Mandelbrot, The Fractal Geometry of Nature (New York: W. H. Freeman, 1982), pp. 1–10.
[2] Felix Hausdorff, 'Dimension und äußeres Maß', Mathematische Annalen, 79 (1918), 157–179.
[3] Yuval Kohavi and Hadar Davdovich, Topological dimensions, Hausdorff dimensions & fractals (Bar-Ilan University, May 2006), p. 5, available at: https://u.cs.biu.ac.il/~megereli/final_topology.pdf (accessed 16 January 2026).
[4] Dimension, Wikipedia, available at: https://en.wikipedia.org/wiki/Dimension (accessed 16 January 2026).
[5] Lecture notes, Fractal Dimension, Stony Brook University, available at: https://www.math.stonybrook.edu/~scott/Book331/Fractal_Dimension.html (accessed 16 January 2026).
[6] Ibid. , p. 5

dimension was invented. It is a way of measuring the “roughness” or complexity of metric spaces and can take non-integer values.[7]

2.1 Similarity Dimension

Now, a question which may arise is, how can we quantify the complexity of self-similar fractals using the Hausdorff dimension? The similarity dimension lets us do this for perfectly self-similar fractals by examining how the number of scaled copies increases as their size decreases[8]. If a fractal can be decomposed into N smaller copies, each scaled by a factor r, then the similarity dimension D is defined by the formula $D = \log(N)/\log(1/r)$. In a nutshell, the formula is really just counting how many smaller copies a fractal has and how much smaller each one is compared to the original. The more pieces there are, or the smaller they are, the higher the dimension. A demonstration of a calculation which depicts the complexity of a fractal using this formula could be with the Sierpinski triangle. First, we must identify what the values of N and r are. We know that they must be 3 and ½ respectively as with each iteration, the triangle is divided into 3 smaller triangles and each smaller triangle is half the side length of the original. If we apply the formula, we get that $D = \log(3)/\log(2)$. This roughly gives us, 0.477/0.301 which is approximately 1.585. Therefore the similarity dimension of the Sierpinski triangle is about 1.585, which is more than a line (1), but less than a plane (2). This shows that the fractal is more complex than a simple shape however it does not fill a 2D area.

2.2 Coastlines and Hausdorff Dimension

“For a regular fractal, if fractal copies/units have no overlapping, the Hausdorff dimension will equal the similarity dimension. […] However, if fractal copies have overlapped parts, the similarity dimension will exceed the dimension of embedding space in value. Thus, the similarity dimension will not equal the Hausdorff dimension.”[9] Effectively, when you take small pieces of a fractal, and they do not cover the same area as another copy, the similarity dimension ≠ Hausdorff dimension. This is because the similarity dimension assumes perfect self-similarity and counts the average number of smaller copies and how much each is scaled down. Whereas the Hausdorff dimension measures the actual space-filling complexity at all scales. So if they do cover the same area, the dimensions would be equivalent, an example of this is that the Hausdorff dimension of the Sierpinski triangle would also be approximately 1.585.

An example of where they are not equal which we are going to explore is coastlines. Imagine a coastline with jagged bays and shores; it is fractal-like, it has roughness at many scales, but it is not perfectly self-similar - no section is an exact copy of another. If we were to calculate the similarity dimension using average scaling, the result would purely be an approximation of the coastline’s complexity. On the other hand, the Hausdorff dimension rigorously and accurately measures how the coastline is not perfectly self-similar, by taking into account its irregularities, and fine details.[10] For example, if we take Britain’s coastline using an average of 5 segments each scaled down by ⅓, the result is approximately 1.465. Using a Hausdorff approach that measures the actual complexity of the coastline at all scales, Mandelbrot estimated that D=1.25[11], which shows that the two dimensions are not always equal. When calculating the similarity dimension for non-self-similar fractals the complexity of the fractal may be over or underestimated, while the Hausdorff dimension provides a more accurate measure of how the space is filled. A huge flaw in the similarity dimension is highlighted as it assumes perfect self-similarity which can misrepresent the complexity of irregular fractals.

3 Hausdorff Dimension Applications

Hausdorff dimension is very useful when we want to analyse real-world patterns that are too irregular for classical geometry. By applying it to natural phenomena, such as plant structures and river networks, we can analyse

[7] Hausdorff dimension, Citizendium, available at: https://en.citizendium.org/wiki/Hausdorff_dimension (accessed 17 January 2026).
[8] K. Falconer, Fractal Geometry: Mathematical Foundations and Applications (3rd edn, Wiley, 2014), p. 15.
[9] Fractal modeling and fractal dimension description of urban morphology, PMC, available at: https://pmc.ncbi.nlm.nih.gov/articles/PMC7597252/ (accessed 17 January 2026).
[10] Nelson, Mark, 12 An Introduction to Fractals [lecture notes], Australian Defence Force Academy, n.d., available at: http://www.ma.adfa.oz.au/~mark/teaching.dir/fractals.html (accessed 16 January 2026).
[11] B.B.Mandelbrot, ‘Statistical self-similarity and fractional dimension’, *How long is the coast of Britain?* ,1967

natural systems such as venation and tributaries. Therefore we can quantify their complexity for science and understanding nature in a way that the topological dimension cannot.

3.1 Plants and Leaves

Fractal dimension has been applied to biology to quantify the complex shapes of leaves, which often show irregular shapes and vein networks that we cannot illustrate using standard Euclidean geometry. Studies have shown that the fractal dimension of leaves is correlated with the complexity of vein networks and their shapes. This allows for differentiation between leaf species via fractals. For example, a study conducted by Sciendo found that the mean fractal dimension of blackberry leaves was around 1.12, which accurately reflects their irregular shape, rather than a completely smooth surface.[12] Researchers have also succeeded in using fractal analysis in order to extract fractal features of both leaf outlines and internal venation for the classification of leaves and comparative studies between species. This shows the practical use of fractal dimension in analysing plant morphology.[13] The application of fractal dimension within leaves and other plants, demonstrates how the Hausdorff dimension can provide a quantitative measure of natural complexity beyond what the standard topological dimension can offer.

3.2 River Networks

Rivers form complex, branching patterns that may resemble fractals. Researchers can use the Hausdorff dimensions to quantify how densely these networks occupy space and the irregularity of their branches, something that the topological dimension cannot do. For instance, the branching of the Amazon River and its tributaries shows a Hausdorff dimension of around 1.7. This reflects the irregularity and extensive branching within the river.[14] This demonstrates the practical applications of Hausdorff dimension within hydrology, flood prediction and environmental planning, which is an excellent exemplar of how fractal measures can capture real-world complexity.

3.3 Urban Patterns

The complexity of human-made structures, such as layouts of cities and road networks may not seem obviously quantifiable as natural structures, like leaves and rivers, however the Hausdorff Dimension can also be applied. Cities often show fractal-like patterns, with streets and blocks forming irregular, branching structures at different scales. The topological dimension of a street network would simply be 1, where roads are treated as simple lines. However, this is flawed, as it ignores how densely they fill space. Using the Hausdorff dimension, researchers are able to quantify the complexity of urban layouts, compare cities and analyse patterns of urban growth and planning.[15] Also, research which was conducted in built-up areas of cities showed that Newcastle has a Hausdorff dimension of around 1.18 whereas Porto has a fractal dimension of around 1.55. This could reflect on a denser, more compact urban form with buildings and streets that are clustered more efficiently in Porto compared to Newcastle. This illustrates how the Hausdorff dimension can capture the irregularities and space-filling complexities within cities - something the topological dimension would fail to do as the city would be treated as simply 1D or 2D.[16]

Conclusion

To conclude, the topological dimension provides a simple measure of dimensionality for regular geometric objects but fails to account for the complexity of fractal structures. Despite the effectiveness of the similarity dimension

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[12] Fractal analysis of leaves: are all leaves self-similar along the cane?, Ekológia (Bratislava), available at: https://doaj.org/article/de3eb32c0d0744d9bc4df0254f0479b3 (accessed 17 January 2026).
[13] Fractal dimension applied to plant identification, Information Sciences, Volume 178, Issue 12 (2008), available at: https://www.sciencedirect.com/science/article/pii/S0020025508000364 (accessed 17 January 2026).
[14] Fractal analysis of river networks: applications in hydrology, Hydrological Sciences Journal, Volume 52, Issue 5, 2007, available at: https://www.tandfonline.com/doi/abs/10.1623/hysj.52.5.1 (accessed 17 January 2026).
[15] M. Batty and P. Longley, Fractal Cities: A Geometry of Form and Function (San Diego, CA and London: Academic Press, 1994).
[16] Fractal dimension of European cities: A comparison of the patterns of built-up areas in the urban core and the peri-urban ring, Cybergeo: European Journal of Geography, available at: https://journals.openedition.org/cybergeo/37243 (accessed 17 January 2026).

for perfectly self-similar fractals, it is inadequate for irregular forms. This is where the Hausdorff dimension can resolve these limitations as it measures how a set fills space across all scales, offering a more precise way of accurately describing the complexity of both artificial and natural structures.