What Is Beauty In Mathematics? (TRM Essay Competition 2026)

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1 Structure and Surprise

In his theory of psychology, Dr. Jung introduced the idea of a 'collective unconscious' containing all the potential psychic material that humanity draws on for its culture and creativity, among other things. We can also see mathematics obeying a similar structure, and in this essay I hope to convince you that *this* is the reason we find mathematics so compelling; poetic notions of pushing the limits of what we *can know* (forever muddied by Gödel) and discovering new connections between disparate fields are endlessly seductive.

The human compulsion to compartmentalise things is a fascinating coping mechanism we employ when attempting to understand an overwhelmingly complex object of study. As such, it is ubiquitous in Mathematics, where concepts quickly become so abstract as to boggle the mind. Mathematicians gainfully employ this compartmentalisation when approaching very daunting conjectures and problems; for a famous (if slightly tired) example, we may turn to the question of Fermat's Last Theorem being solved by way of the Taniyama-Shimura-Weil conjecture, reducing the problem from a Diophantine nightmare to a problem concerning elliptic curves and their modularity. In such a way, Sir Andrew Wiles was able to ingeniously construct the missing proof for Fermat's Last Theorem 350 years after it was conjectured.

The compartments or 'fields' of mathematics, while appearing as disparate islands are in fact spanned by more bridges than Königsberg (graph theorists, I hope you're pleased with yourselves). As such, we can investigate these (sometimes unexpected) links, and this is where we uncover the elusive 'beauty'. World's longest preamble now out the way, it's worth trying to give some description of what mathematicians mean by 'beauty'. More or less, beauty in mathematics is the best thing ever or entirely pointless (depending on your point of view); nothing more and nothing less. It's the pleasure of a long argument resolving itself in a neat answer and unexpected solutions to tricky problems obtained through innovative means. That being said, let's take a look for oursleves and see what this is really all about.

2 'Beautiful' Mathematics

Now we can finally take a look at some examples:

1. Euler's Formula
2. The Basel Problem
3. The Gamma Function
4. Fractional Calculus

2.1 Euler's Formula

The most famous in all mathematics, and not the only one to go by this name (that would be far too simple), Euler's formula states:

$$e^{i\pi} + 1 = 0$$

It can be derived by means of power series (which will follow presently), but first it's important to appreciate why *this* equation above all others is considered 'beautiful' if we are to have any intuition about what makes beautiful mathematics. Here, the answer is straightforward - seeing five of the most fundamentally important constants in mathematics (two of which happen to be 'transcendental', and one 'imaginary') united in one line of elegance is what makes Euler's formula stand head and shoulders above the crowd. Now, the derivation - consider the following Taylor series:

$$e^z = 1 + \frac{z}{1!} + \frac{z^2}{2!} + ...$$

And let $z = ix$ such that:

$$e^{ix} = 1 + ix + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + ... = 1 + ix - \frac{x^2}{2!} - i\frac{x^3}{3!} + ...$$

Now, assuming that this power series is absolutely convergent for all complex numbers z, (i.e the sum of the magnitudes of the terms in the series converges), we can rearrange the terms in this series without affecting its value. We use this fact to group the imaginary numbers and real numbers separately:

$$e^{ix} = (1 - \frac{x^2}{2!} + \frac{x^4}{4!} + ...) + i(x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...)$$

Which we can now recognise as the Taylor series for $cos(x)$ and $sin(x)$ respectively, so we have the result:

$$e^{ix} = cos(x) + isin(x).$$

Now letting $x = \pi$ and adding one to both sides, we finally have the formula in its famous form:

$$e^{i\pi} + 1 = 0$$

2.2 The Basel Problem

Posed by Mengoli in 1650, the Basel Problem asked what the sum of the reciprocals of the squares of the natural numbers is (what a mouthful!); it withstood attack from mathematicians for almost a century until Euler solved it (not quite rigourously) in 1734.

His method was remarkable for how it exploits the strange connection between polynomials, the sine function and the infinite sum in question. First, consider the Taylor series for $sin(x)$:

$$sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} - ...$$

Divide by x:

$$\frac{sin(x)}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - ...$$

Factorising:

$$\frac{sin(x)}{x} = \Big(1 + \frac{x}{\pi}\Big)\Big(1 - \frac{x}{\pi}\Big)\Big(1 + \frac{x}{2\pi}\Big)\Big(1 - \frac{x}{2\pi}\Big)... = \Big(1 - \frac{x^2}{\pi^2}\Big)\Big(1 - \frac{x^2}{4\pi^2}\Big)...$$

We notice denominators which suspiciously match the series we are trying to evaluate. Now consider just the x^2 coefficient:

$$-\Big(\frac{1}{\pi^2} + \frac{1}{4\pi^2} + \frac{1}{9\pi^2} + ...\Big) = -\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

From the original series, we know the x^2 coefficient must also equal $-\frac{1}{6}$;

$$-\frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2} = -\frac{1}{6}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^2} = \frac{\pi^2}{6}$$

How elegant!

2.3 The Gamma Function

An interesting (I hope I am not alone in thinking) feature of many Taylor series, including the expansions for $cos(x)$ and $sin(x)$ above, is a variable raised to the n-th power in the numerator over $n!$ in the denominator. Upon closer inspection after I first noticed this, I realised the pattern was there because, after n successive differentiations (as Taylor series are built), the n-th term of such a series becomes:

$$n(n-1)(n-2)...(2)(1)\frac{kx^{n-n}}{n!} = n!\frac{k}{n!} = k$$

So this pattern reflects how repeated differentiation neatly cancels n-th degree powers with the product of the first n natural numbers (when $n \in N$). This relationship is a major indicator that there is some kind of link between the factorial function and calculus - after all, "there are no accidents" (thank you Dr Freud/Master Oogway)!

We'll return to calculus later. For now, what happens when we generalise the factorial function to define it for non-natural number inputs? As luck would have it, mathematicians have long since tackled this particular generalisation (of course they have!). The solution, courtesy once again of Mr Euler, is the called the gamma function. The gamma function allows us to make sense of any complex input to the gamma function (except negative integers). We define:

$$\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}dt = (n-1)!$$

For any complex z which isn't a negative integer. Now would you look at that - Houston, we have calculus! Another coincidence between the two surely must now inspire us to find ways to make further use of the connection.

2.4 Fractional Calculus

Mathematicians are infamously lazy. One day, so the story goes, a certain Mr Leibniz was *so* lazy he wanted to take just an itsy bit of a integral. Just a smidge. He wrote about the idea in a letter to another mathematician, l'Hopital (of eponymous rule fame), but in classic style never entirely found a solution to his problem. Thankfully, in the stretch of time between us and Gottfried 'the layabout' Leibniz, someone *did* find a solution, which now follows. How do we even make sense of a fractional integral? What does that mean? A good place to start is with Cauchy's formula for repeated integration:

$$(J^n f)(x) = \frac{1}{(n-1)!}\int_0^x (x-t)^{n-1}f(t)dt$$

Now if we can re-purpose this for fractions (which seems possible, as fractions are the multiplicative inverses), we're golden. Let's notice this formula currently is only defined for $n \in N$, so we need to expand the domain. Let us *also* recall we can generalise the factorial function to any complex input (including fractions) using the gamma function. How convenient. Since $\Gamma(z) = (n-1)!$,

$$(J^z f)(x) = \frac{1}{\Gamma(z)}\int_0^x (x-t)^{z-1}f(t)dt.$$

Huzzah! Having determined a method for finding the partial integral of an expression (this particular integral is called the Riemann-Liouville integral), we can naively claim victory and push some of the more subtle issues with this method (for instance, that many irrational and non-half-integer inputs don't have a closed form expression, so there are many fractional integrals we can't easily compute) under the proverbial rug. An example of the Riemann-Liouville integral in action may look as follows:

$$
{}_{\frac{1}{2}}\int_0^x 2tdt^{\frac{1}{2}} = \frac{1}{\Gamma(\frac{1}{2})}\int_0^x (x-t)^{\frac{1}{2}-1}2tdt = \frac{1}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}2tdt
$$

$$
= \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}}
$$

Which yields us a power halfway between one and two, as we would hope. It turns out defining a fractional derivative can actually be done making use of the Riemann-Liouville integral; if we first take the some fraction of an integral to achieve our desired power (in the case of polynomials) and then apply one or several regular derivatives, we achieve our desired result. To illuminate the argument, here is another example where we find the $\frac{1}{2}$ derivative of t^2.

$$
\left(\frac{d}{dt}\right)^{\frac{1}{2}}(t^2) = \frac{d}{dt}\left(\frac{1}{\sqrt{\pi}}\int_0^x (x-t)^{-\frac{1}{2}}t^2dt\right) = \frac{d}{dt}\left(\frac{16}{15\sqrt{\pi}}x^{\frac{5}{2}}\right)
$$

$$
\frac{d}{dt}\left(\frac{16}{15\sqrt{\pi}}x^{\frac{5}{2}}\right) = \frac{80}{30\sqrt{\pi}}x^{\frac{3}{2}} = \frac{8}{3\sqrt{\pi}}x^{\frac{3}{2}}
$$

So we can sleep easy at night knowing the half-integral of $2t$ is equivalent to the half-derivative of t^2.

So we conclude this exploration of the exponential function and one of its connections to calculus, but know that given any area of inquiry one would find a similarly fascinating path; such is the beauty of the subject.

3 Conclusion

So we've seen how maths is densely wrapped round itself (like a pig in its blanket), however the most mysterious aspect of mathematics is surely its uncanny ability to describe the phenomena of the natural world, which seems to hint at some numinous link between mathematics and the structure of the universe. Indeed, this is probably what inspires the most mathematicians (directly or indirectly) to pursue the subject. One might joke that only a supernatural force could draw healthy, happy human beings to this subject, but in reality you could argue the jokester isn't far from the mark; it would appear in many cases that atheist mathematicians have substituted a god or gods in of traditional sense for mathematics instead. They devote their lives to its study like theologians, they work to contribute something to its glory like parishioners donating for a new church and they hold the greatest among them up on vast pedestals as if they were prophets.

This should speak to the power of the beauty of mathematics. Despite being an elusive thing, its consistent ability to show up where you least expect it is what makes it so awe-inspiring. Mathematics, like an ogre, is an onion of very many layers (thank you Shrek) turgid throughout with the pickling juice of a million interlinked objects. That's the last pun from me (for the best I think!). Thank you for reading.