What Makes a Number a Number?

“Numbers” you say?

“Hey John, why the long face?”

“Oh hi Andy, it's nothing. It's just that I want to take a quick nap under this tree, but I can't seem to fall asleep. And I’ve got all these sheep, but I don’t know how many there are or how I’d even figure that out.”

“John, you won't believe me, but I have the exact thing to solve all of your problems”

“Really?”

“Yep, it's this new thing I invented called counting. It's when you assign numbers, in a specific order, to objects to represent quantity. And, if you apply it to your sheep, you'll be asleep in no time. ”

John pauses.

“But…what is a number?”

John's question (from this totally real excerpt from the first conversation about numbers that I didn't just make up) proceeded to stump Andy, and anyone bold enough to ask that same question for millennia, not because they couldn't find a satisfactory definition, but because they found multiple – and they all yielded the same mathematics.

Alright, so maybe this wasn't the first conversation about numbers (but, it still could've occurred at some point). Numbers weren't just invented by a single person. And, contrary to popular belief, numbers (and the subsequent math) weren't invented to simply torture us. Ancient civilizations needed a practical way to track data, such as resources, inventory, trade, etc., especially for when the quantity of data exceeded 10 (the total number of fingers on our hands). To supplement this, they used tally marks – vertical strokes on bones, wood, or stone – for counting and recording. These tally marks were the beginnings of what we now call the “natural” numbers or the “counting” numbers.

The system starts to crack

Now, for a while, the naturals were adequate for our purposes. But, they'd soon be found to not be sufficient. What happens when a farmer sells all of their 9 cows – how do you represent that on a tally? Or when 20 of your crops grow to their expected height of 4 feet, but one is 2 feet tall – should that be counted as 21 or left out of the tally? Or when a debtor, who owes you 300 silver pieces, gives you 305 – how do future transactions work? Today, the answers to these questions seem obvious: the farmer now has 0 cows, that one crop is counted as a half, and the debtor now has a negative balance. However, for earlier humans, it wasn’t that simple. Fractions, negatives, even zero hadn’t yet been introduced. The system, and its capabilities, was limited. It needed an upgrade (or punnily, it needed an addition). Enter: the integers.

The integers, the set containing all of the counting numbers, their negative counterparts, and the concept of zero would help rectify these situations. But, what would this upgrade mean for our understanding of numbers? Before, numbers were understood through concrete objects – a utilitarian purpose. Now, they were becoming more abstract – physicists, chemists, philosophers were using them to describe the underlying workings of the universe – numbers as merely symbols for quantity wasn't cutting it anymore. What did “1” really mean? Can you prove “4”? Explain “9” without reference to any other number or counting physical objects.

Axioms to the rescue

Among the most notable attempts to answer these questions was: Giuseppe Peano. Giuseppe, a 19th century mathematician and linguist, believed ordinary language was full of too many ambiguities. He thought that math should be developed formally, to avoid such mistakes and create a precise, universal language – starting from its very core: numbers. He sought to accomplish this, and in 1889, he published a set of axioms – statements assumed to be true but without proof – that rigorously and formally defined the natural numbers. The axioms can be stated as follows:

Axiom 1: 0 is a natural number ($0 \in \mathbb{N}$).

Axiom 2: Every natural number n has a successor, $S(n)$, which is also a natural number.

Axiom 3: 0 is not the successor of any natural number (i.e., $S(n) \neq 0$ for all n).

Axiom 4: If the successors of two numbers are equal, the numbers themselves are equal (i.e., if $S(m) = S(n)$, then $m = n$).

Axiom 5 (the inductive step): If a set contains 0 and the successor of every number in that set, then the set contains all natural numbers.

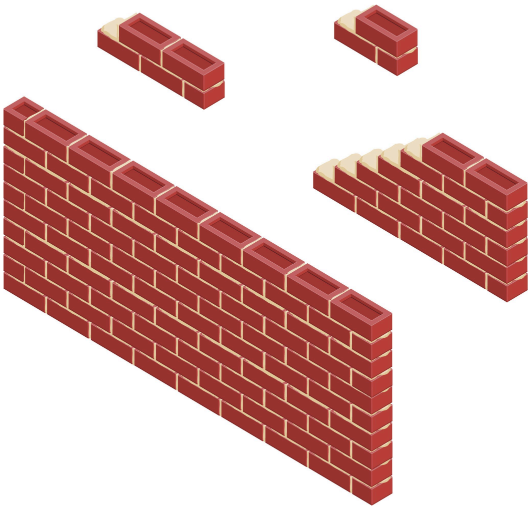
Notice what Peano did and did not say. He did not define numbers by simply pointing at an object and saying “one”. He did not define them by their purposes e.g., representing quantity. Instead, he defined them by their structure – an infinite chain of successor functions wherein if you have a starting point (represented by the symbol '0', for example) and a way to get to the ‘next’ (e.g., by use of the successor function, $S(n)$) without accidentally looping back to 0, then voilà: you’ve got the natural numbers. Elegant, amirite?

Nothing makes something…

Peano's axioms were a giant leap, but their logical consistency is why they’ve remained relatively unchanged and widely adopted by mathematicians. But Peano wasn’t the only one trying to define numbers. Recall what you were doing at age 20. Studying in college? Finding yourself? #YOLOing? All fairly standard, but, I’m pretty certain (though not 100%) you weren’t publishing papers that changed how entire fields think about a topic. Yet, such is the case with John Von Neumann – a quirky fella, he did his best work in noisy, chaotic environments. Von Neumann had a somewhat bold idea: Let's show that numbers (and more generally, all mathematical objects) can be defined as purely set-theoretic entities. And in 1923 he did just that (at least for the natural numbers).

Consider a set. It's empty. We'll call it (and you might want to sit down for this) an 'empty set'. This is our building block. From this building block, we can make a house (the natural numbers). "How?" you may ask. By use of our cement - the successor function, $S(n)$. This function is an operation that takes the set of all smaller numbers, $S(n) = n \cup \{n\}$
Every number, n, becomes the set of previous numbers i.e., a wall in our house is actually a collection of previous bricks.

$0 = \{\} = \emptyset$. [The empty set, the starting point]
$1 = \{0\} = \{\emptyset\}$
$2 = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$
$3 = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$
$4 = \{0, 1, 2, 3\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}\}$



In Von Neumann's world – and now ours – a number isn't an independent entity, but the result of a recursive construction from smaller numbers. Start from nothing, we have nothing. But that nothing we have is something we now have. And that something we now have plus that nothing we had earlier is now more things we have. Mind-bending, I know. (Notice how both Von Neumann and Peano made use of the 'Successor function', this isn't a coincidence. They're related. Can you explain how?).

A cat is an animal that cats.

Believe it or not, but there was yet another attempt at defining numbers – as I said earlier, a lot of people dedicated effort to this seemingly trivial question. Of particular interest to us, though, is Alonzo Church's work. Preferring that the foundations for mathematics be based on functions rather than set theory, Alonzo, building on his lambda calculus work, asked: 'What if a number isn't a set or an axiom, but a function?' . In 1932/33, he answered that question with the development of his now-famous 'Church numerals'.

In this system, a number n can be thought of as an application of a function f to an argument x exactly n times.

0: $\lambda f. \lambda x. x$
1: $\lambda f. \lambda x. fx$
2: $\lambda f. \lambda x. f(fx)$
3: $\lambda f. \lambda x. f(f(fx))$
4: $\lambda f. \lambda x. f(f(f(fx)))$

Now at first glance this may seem circular, defining a number ('1' for example), using the very notion of the number (e.g., '1 time') you're defining, in your definition. That's like responding to the question 'What is a cat?' by saying "A cat is an animal that is a cat". But, here's the kicker: in lambda calculus, the trick is that "one time" isn't actually part of the definition. It's more of an after-the-fact English interpretation. The actual definition e.g. $\lambda f. \lambda x. fx$ doesn't mention numbers. We've really just assigned "1" to this function. So, unlike other frameworks, a number isn't a noun, but a verb or action (that action being the function being applied), or, using our previous analogy, a cat is an animal that cats. Now you may wonder what exactly is the function being applied i.e. what does ‘catting’ look like. And here’s the key part: the function being applied doesn’t “do” anything specific, it’s a placeholder (which, I might add, is very similar to catting. #dogperson).

Reconciling bone marks, functions and everything in-between

Tally marks on bones, integers, sets, functions, so many definitions of numbers, and they’re all different, which raises two questions: Which one is the true definition? And, more importantly, how can the foundations of the (supposedly) most coherent field be…inconsistent?

Well, the thing is: they’re **not** inconsistent. They’re different, yes, but despite the differing models, they all result in the same math. This is because a number isn’t defined by its essence, or by its ‘what’. A number is defined by its adherence to the rules and roles in a specific system, a system governed by rules like those in the Peano axioms. Zero, successor, no loops, induction – these are the rules, and they don’t care what a number *is*, only how they relate. So, when Von Neumann defined ‘1’ as $\{\emptyset\}$, and when Church defined ‘1’ as $\lambda f. \lambda x. fx$, they were both correct, because they satisfied the same rules. And these rules? These are what make a number a number.
Quite **E**nough **D**one.

As for John and Andy

John, luckily, did end up falling asleep (apparently, Andy’s ‘counting sheep’ suggestion was a game changer). I wish I could say the same for Andy, but he spent the rest of his time trying to answer John’s question and, unfortunately, he never reached the same realization we did. And, if you’re wondering, there were $S(S(S(S(S(S(S(S(S(S(S(S(0))))))))))))$ sheep.