When primes walk randomly

By Shriya Rayaraddi

Introduction

Imagine a child standing beneath a lamppost on an empty street. Every second, she flips a fair coin: heads, she steps one metre to the right; tails, one metre to the left.

Over time, however, something unexpected happens. Despite the coin being fair, she does not stay near the lamppost. The walk has no bias, yet large deviations naturally appear.

Mathematics provides a precise explanation. Each step is either +1 or −1, with average value zero. But the average alone is misleading. In reality, small random steps accumulate, causing the position to drift away from the start. The key is not the average, but the spread. As the number of steps increases, the spread grows - not linearly, but like the square root of the number of steps. Randomness, while unpredictable, still follows a precise pattern. This slower growth reflects a key feature of randomness: even though outcomes vary, they do so in a controlled way.

At first, this seems unrelated to prime numbers. Primes are deterministic, yet their distribution appears irregular, sometimes clustering and sometimes leaving large gaps.

In 1859, Bernhard Riemann proposed a striking idea: these irregularities may follow a law similar to a random walk. His conjecture, the Riemann Hypothesis, suggests fluctuations in the distribution of primes are bounded roughly by a square-root scale. Understanding this leads to unexpected ideas (which we explore now!): a function of complex numbers encoding primes, an arithmetic function assigning signs to integers, and a deep link between randomness and number theory.

I Random walks and the growth of fluctuations

Let us describe the child's movement more precisely. Each step can be represented by a random variable:

$$X_i = \begin{cases} +1 \text{ with probability } 1/2 \\ -1 \text{ with probability } 1/2 \end{cases}$$

where Xi denotes the i-th coin flip. After 'n' steps, the child's position is the sum

$$S_n = X_1 + X_2 + \cdots + X_n$$

Because the coin is fair, the expected value of a single step is

$$E[X_i] = (+1) \times \frac{1}{2} + (-1) \times \frac{1}{2} = 0$$

Using the linearity of expectation,

$$E[S_n] = E[X_1] + E[X_2] + \cdots + E[X_n] = 0$$

On average, the child remains at the lamppost. But expectation alone hides the real behaviour of the walk. A typical walk does not stay near the origin; it drifts away. To understand this, we must measure the size of the fluctuations rather than the expected value of a single step, which is done using variance. Since Xi^2 = 1 for every step,

$$\text{Var}(X_i) = E[X_i^2] - (E[X_i])^2 = 1 - 0 = 1$$

The steps are independent, so variances add:

$$\text{Var}(S_n) = \text{Var}(X_1) + \text{Var}(X_2) + \dots + \text{Var}(X_n)$$

The standard deviation, which measures the typical distance from the origin, is therefore

$$\sqrt{\text{Var}(S_n)} = \sqrt{n}$$

This reveals a fundamental principle of randomness. When independent influences accumulate without bias, fluctuations grow on the square-root scale. Doubling the number of steps doubles the variance, but only increases the typical distance by a factor of √2, so randomness spreads slowly. Even after ‘n’ steps, the walk typically lies only about √n units from where it began.

Such behaviour appears widely in science, from particle motion to financial markets, where small random effects combine to produce controlled fluctuations.

This raises an intriguing question. Prime numbers are deterministic; they are rigid objects defined by divisibility, yet their distribution appears irregular. The number of primes below a large value ‘x’ fluctuates around its predicted value, but how large are these fluctuations?

Remarkably, one of the deepest conjectures in mathematics suggests the answer is exactly the same scale that governs a random walk. The irregularities of the primes, it turns out, may grow in a way strikingly similar to the wandering of the child beneath the lamppost. To explore this, mathematicians introduce an arithmetic function that translates multiplicative structure into a sequence of signs.

II The Möbius Function and arithmetic randomness

To translate questions about primes into something more “dynamic,” mathematicians use a function that assigns simple numerical signals to integers. This is the Möbius function, written as μ(n).

Rather than measuring size, μ(n) reflects the structure of a number's prime factors. If a number includes any repeated prime factor, it is assigned 0. Otherwise, it is given either +1 or −1 depending on whether it has an even or odd number of distinct prime factors. In this way, each integer contributes a kind of "positive" or "negative" signal, allowing patterns to emerge when many values are combined.

$$\mu(n) = \begin{cases} 1 & \text{if } n \text{ has an even number of distinct prime factors} \\ -1 & \text{if } n \text{ has an odd number of distinct prime factors} \\ 0 & \text{if } n \text{ is divisible by the square of a prime} \end{cases}$$

This measures the cumulative balance between positive and negative contributions. If a bias existed, M(x) would grow proportionally to x.

To illustrate, we can compute a small example. Consider the values of μ(n) for $n \leq 8$:

- $\mu(1) = 1$
- $\mu(2) = -1$
- $\mu(3) = -1$
- $\mu(4) = 0$ (since $4 = 2^2$ contains a squared prime)
- $\mu(5) = -1$
- $\mu(6) = +1$
- $\mu(7) = -1$
- $\mu(8) = 0$ (since $8 = 2^3$ contains a squared prime)

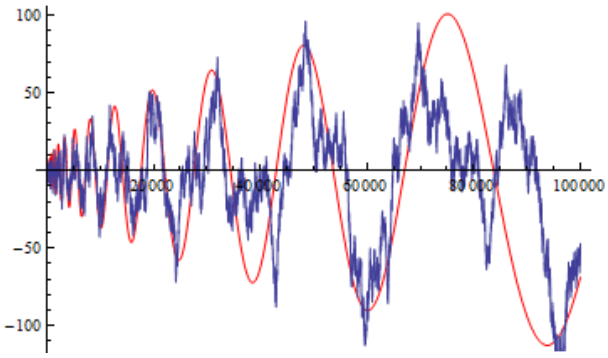
Adding these values gives: $M(8) = 1 - 1 - 1 + 0 - 1 + 1 - 1 + 0 = -2$. Even in this small example, the sum fluctuates irregularly rather than growing steadily, giving an early glimpse of the "random-like" behaviour that becomes more pronounced for larger x.

At first glance, this definition appears quite artificial, but it captures how, in arithmetic, the way integers are built multiplicatively from primes. In fact, the Möbius function is multiplicative, meaning that whenever two integers a and b share no common factors: $\mu(ab) = \mu(a)\mu(b)$

This property allows it to act as a kind of arithmetic filter, separating the structure of integers into positive and negative contributions. To study how these signs accumulate, mathematicians consider the Mertens function:

$$M(x) = \sum_{n \leq x} \mu(n)$$

This sum measures the cumulative balance between +1 and −1 values among the integers up to x. If the signs showed a systematic bias, the sum would drift steadily away from zero. For instance, if positive values dominated, M(x) would grow roughly proportionally to 'x'.



Source - Math stack exchange, Mertens function

(Numerical behaviour of M(x) has been extensively studied; see discussions of the Mertens function in analytic number theory.)

Numerical evidence suggests this behaviour persists: M(x) oscillates around zero with no clear long-term trend, resembling a random walk. This leads to a natural heuristic. If the non-zero values of $\mu(n)$ behaved like independent ± 1 steps, then by the random walk model in Section I, the sum after x terms would typically be of size $\sqrt{x}$. In other words, we would expect

$$M(x) \approx \sqrt{x}$$

This is not a proof, since the Möbius values are not independent, but it provides a compelling heuristic explanation for why √x emerges as the natural scale of fluctuation. In analytic number theory, this viewpoint, often referred to as the Möbius randomness principle, reflects a more general notion: randomness can direct our comprehension of deterministic structures. The anomalies in the distribution of primes would be closely controlled if M(x) is in fact bounded by about the square-root scale.

In fact, a profound connection discovered by Bernhard Riemann shows that the behaviour of the Möbius sums is governed by a single analytic object: the Riemann Zeta Function. Its structure leads directly to the conjecture now known as the Riemann Hypothesis. To see how primes are encoded within a function of complex numbers, we now look at the remarkable identity that first revealed this hidden structure.

III The function that encodes the primes

To understand how the behaviour of the Möbius sums relates to prime numbers, mathematicians study a remarkable function introduced by Bernhard Riemann in 1859: the Riemann Zeta Function.

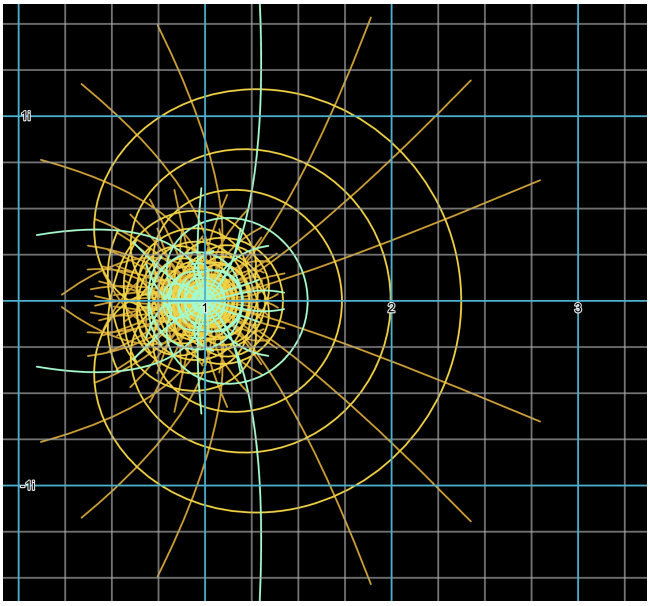
For a complex number s with real part greater than 1, the function is defined by the infinite series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}$$

At first sight, this expression seems unrelated to primes. It simply sums reciprocal powers of the integers. But in this series, there lies one of the most beautiful discoveries in mathematics. More than a century earlier, mathematician Leonhard Euler proved that the same function can also be written as an infinite product over the primes:

$$\zeta(s) = \prod_{p \text{ prime}} \frac{1}{1 - p^{-s}}$$

This identity, known as the Euler product, reveals a remarkable phenomenon. The primes are not merely scattered through the integers; they are encoded directly inside the structure of the zeta function. Every prime contributes one factor to the product. If a prime were removed, the function itself would change.



The reason this works is the fundamental theorem of arithmetic: every integer can be written uniquely as a product of primes. When the product above is expanded, the terms generated correspond exactly to the reciprocals 1/n^s appearing in the series definition. In this way, multiplicative structure in the integers becomes analytic structure in the function. The zeta function, therefore, acts like a bridge between the discrete world of prime numbers and the continuous world of complex analysis.

LEFT: Riemann Zeta function on Desmos

Riemann's key insight was to study what happens when the function is extended beyond the region where the series converges. Through a process called analytic continuation, the zeta

function can be defined for almost all complex numbers ‘s’. Once extended, it exhibits a remarkable pattern of zeros, which are points where

$$\zeta(s) = 0$$

Some occur at the negative even integers (the trivial zeros), while the non-trivial zeros lie in the critical strip $0 < \mathrm{Re}(s) < 1$. Their location controls the irregularities in the distribution of primes.

Riemann conjectured that every non-trivial zero has a real part exactly equal to $\mathrm{Re}(s) = ½$, a statement now known as the Riemann Hypothesis. Although the hypothesis concerns a function of complex numbers, its consequences reach deep into number theory. If true, the error in estimating the number of primes below a large number ‘x’ would remain bounded on the same square-root scale that appears in random walks.

Even more strikingly, the hypothesis also implies a bound on the Möbius sums that were discussed earlier on:

$$M(x) = \sum_{n \leq x} \mu(n)$$

Under the Riemann Hypothesis, these sums cannot grow much larger than about $\sqrt{x}$. In other words, the signs produced by the Möbius function would cancel almost perfectly, fluctuating like a random walk.

Here, an unexpected connection emerges! The wandering of a child flipping coins, the arithmetic signs determined by prime factorisation, and the 0s of a complex function are all governed by the same phenomenon, that is, fluctuations that grow like the square root of their scale. The Riemann

Hypothesis thus suggests that beneath the rigid structure of the integers lies a hidden statistical balance.

IV Randomness might be the key!

At first glance, prime numbers seem unpredictable. They become less frequent as numbers grow, yet their exact positions follow no clear pattern. However, the Riemann Hypothesis suggests something striking: primes may behave as if governed by randomness.

One way to see this is through the Möbius function. Its cumulative sum rises and falls irregularly, producing a jagged path resembling a random walk. This similarity is more than just a visual one. If the values behaved like independent ±1 steps, their sum would typically grow on the square-root scale. Controlling this growth is closely tied to the Riemann Hypothesis. This creates a paradox: primes are deterministic, yet resemble random outcomes. They seem to lie at the boundary between order and chance…

To conclude, in a random walk, fluctuations of size $\sqrt{n}$ arise from the accumulation of many independent steps. The Riemann Hypothesis suggests that the primes, despite being completely deterministic, fluctuate on exactly the same scale.

What I find most striking is not just this numerical coincidence, but what it suggests about mathematics itself. Primes follow exact rules, yet collectively imitate randomness so well that probability becomes the natural way to describe them.

This raises a deeper question: is randomness truly separate from structure, or simply what complex structure looks like?

In this sense, the Riemann Hypothesis may reflect a limit. Not just on primes, but on how much order we can identify. Sometimes, patterns can appear indistinguishable from chance. And so the child beneath the lamppost, wandering unpredictably through coin flips, may not be so different from the primes after all :)