Why are there so many ways of multiplying vectors?

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1 Introduction

When I was preparing for my further maths mock, I eventually got to the section about the dot product of vectors. This essay started as a simple thought about why we use the dot notation ($\vec{u} \cdot \vec{v}$) to represent this instead of how we typically show multiplication with numbers (ab). After multiple days of research, I had learnt about another way of multiplying vectors, as well as the basics of a field of maths, called geometric algebra, which I have compiled the core of here.

2 Why not just expand it?

One of the first thoughts that I had was "What if I just expanded the brackets out?" Doing so with two 3D vectors gave me this:

$$\begin{aligned}\left(u_1\hat{\imath} + u_2\hat{\jmath} + u_3\hat{k}\right)\left(v_1\hat{\imath} + v_2\hat{\jmath} + v_3\hat{k}\right) =& u_1v_1\hat{\imath}\hat{\imath} + u_1v_2\hat{\imath}\hat{\jmath} + u_1v_3\hat{\imath}\hat{k} + \\ & u_2v_1\hat{\jmath}\hat{\imath} + u_2v_2\hat{\jmath}\hat{\jmath} + u_2v_3\hat{\jmath}\hat{k} + \\ & u_3v_1\hat{k}\hat{\imath} + u_3v_2\hat{k}\hat{\jmath} + u_3v_3\hat{k}\hat{k}\end{aligned}$$

This simply raised a lot more questions, and also didn't even help us further, as we now have the products of the different basis vectors. Let's try this with 2D vectors and see if we get anything simpler:

$$\left(u_1\hat{\imath} + u_2\hat{\jmath}\right)\left(v_1\hat{\imath} + v_2\hat{\jmath}\right) = u_1v_1\hat{\imath}\hat{\imath} + u_1v_2\hat{\imath}\hat{\jmath} + u_2v_1\hat{\jmath}\hat{\imath} + u_2v_2\hat{\jmath}\hat{\jmath}$$

Nope! Still something that I have no idea how to comprehend. Here are some of the questions that I had about this:

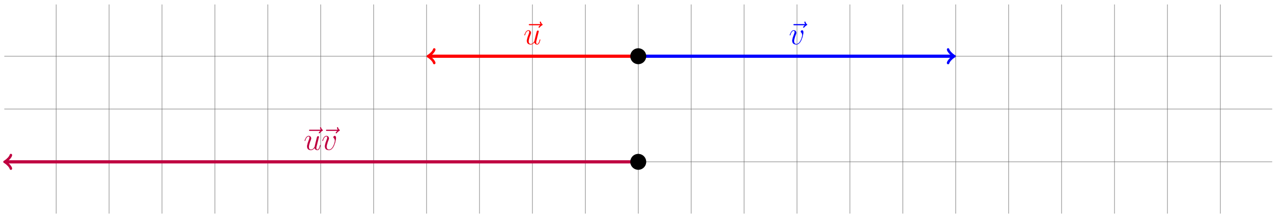
- What kind of quantity is the product of two vectors?
- How can we represent this geometrically?
- Does this have any real world uses?

3 Where do we go from here?

The first thing that we can do to help simplify this further is to try and remove some of the vector multiples. The first thing that came to my mind to try and remove were the $\hat{\imath}\hat{\imath}, \hat{\jmath}\hat{\jmath}, \hat{k}\hat{k}$. So how do we do that? Now, I could just tell you how, but that wouldn't help with your actual understanding, so to help us get a feel for what we are actually doing, we need to move into just one dimension for a small amount of time.

4 Multiplying Vectors in One Dimension

Imagine we only have one dimension. How would we multiply two vectors, $\vec{u}$ and $\vec{v}$ in this space? The first think that we can try is to treat one-dimensional vectors as numbers, and just take multiplication from them, as shown in the below example:



The issue in doing this is that it makes it necessary for one direction to be positive and the other negative, and each direction acts differently. For example, when you multiply two right pointing vectors in this way, the result points right. But when you multiply two left pointing vectors, the result still points right. Apparently mathematicians don't want this, so we need another way.

It turns out that it is impossible to have an operation on two vectors that returns another vector without singling out one direction, but who said that the product of two vectors has to be a vector? We can have our product be a scalar, which is exactly what we do!

Let's now try to come up with a definition for the product of parallel vectors that returns a scalar. The first definition we can try is to simply multiply the magnitudes as in $\|\vec{u}\| \|\vec{v}\|$. The problem here is that this definition isn't associative, meaning $\vec{u}(\vec{v}\vec{w}) \neq (\vec{u}\vec{v})\vec{w}$. We can prove this as follows:

Proof. Let $\vec{u} = \hat{i}, \vec{v} = 2\hat{i}, \vec{w} = -2\hat{i}$

$$\begin{aligned}
\vec{u}(\vec{v}\vec{w}) &= \vec{u}(\|\vec{v}\| \|\vec{w}\|) \\
&= \vec{u}((2)(2)) \\
&= 4\vec{u} \\
&= 4\hat{i} \\
(\vec{u}\vec{v})\vec{w} &= (\|\vec{u}\| \|\vec{v}\|)\vec{w} \\
&= ((1)(2))\vec{w} \\
&= 2\vec{w} \\
&= -4\hat{i}
\end{aligned}$$

Because $\vec{u}(\vec{v}\vec{w}) \neq (\vec{u}\vec{v})\vec{w}$, this definition is not associative. ■

This issue occurs because the sign disappears when we take the magnitudes, so any direction information is removed. The solution for this is surprisingly simple. Because we are in just one dimension, all the vectors are parallel, so we add a minus sign when the vectors point in different directions. This solves the associativity problem by making sure that information about direction is preserved. If we consider the previous problem:

$$\begin{aligned}
\vec{u}(\vec{v}\vec{w}) &= \vec{u}(-\|\vec{v}\|\,\|\vec{w}\|)\\
&= \vec{u}(-(2)(2))\\
&= -4\vec{u}\\
&= -4\hat{i}\\
(\vec{u}\vec{v})\vec{w} &= (\|\vec{u}\|\,\|\vec{v}\|)\vec{w}\\
&= ((1)(2))\vec{w}\\
&= 2\vec{w}\\
&= -4\hat{i}
\end{aligned}$$

Now that we have sorted associativity out, we can make this our definition for multiplying parallel vectors:

$$\vec{u}\vec{v} = \begin{cases} \|\vec{u}\|\,\|\vec{v}\| & \text{if same direction} \\ -\|\vec{u}\|\,\|\vec{v}\| & \text{if opposite direction} \end{cases}$$

From this definition, let's see what happens if we square a vector. Because a vector is parallel to itself and points in the same direction as itself, we get this identity:

$$\vec{v}^2 = \|\vec{v}\|^2$$

This is incredibly helpful because it allows us to simplify the $\hat{i}\hat{i}, \hat{j}\hat{j}, \hat{k}\hat{k}$ parts of our arbitrary product that we expanded at the beginning. Because the magnitudes of these basis vectors are 1, the square of these vectors is also 1. Let's now see how this makes what we are dealing with easier:

For 2D vectors the product can now be written as:

$$\vec{u}\vec{v} = u_1v_1 + u_2v_2 + u_1v_2\hat{i}\hat{j} + u_2v_1\hat{j}\hat{i}$$

And for 3D vectors we can write it as:

$$\begin{aligned}
\vec{u}\vec{v} =& u_1v_1 + u_2v_2 + u_3v_3 +\\
& u_1v_2\hat{i}\hat{j} + u_2v_1\hat{j}\hat{i} +\\
& u_2v_3\hat{j}\hat{k} + u_3v_2\hat{k}\hat{j} +\\
& u_3v_1\hat{k}\hat{i} + u_1v_3\hat{i}\hat{k}
\end{aligned}$$

If you think the first part looks familiar, that's because it is! The first part of both of these products is the dot product of the two vectors! Note that I have never used the dot product in deriving this, and that it has just come out to us.

5 Multiplying Perpendicular Vectors

We have successfully found a way to multiply two parallel vectors, allowing us to simplify part of our products. For the rest of it, we need to look at multiplying perpendicular vectors, which is what our basis vectors are. This will help us to determine what $\hat{i}\hat{j}$ and the other ones actually are. We cannot use the dot product, because then it would always be 0 due to basis vectors being perpendicular, and that would be boring and pointless. Instead, we want something that we can actually interpret in a meaningful way.

The tool that is used in geometric algebra to do this is a bivector.

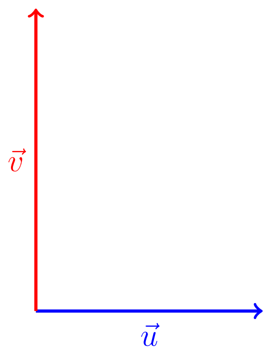
5.1 What is a bivector?

To understand what a bivector is, we first have to think of what a vector is. We can think of a vector as being an oriented line segment. This means that it has three properties: the line that it is a segment of, its length (magnitude), and the orientation, which is the direction it points in (forwards or backwards on the line).

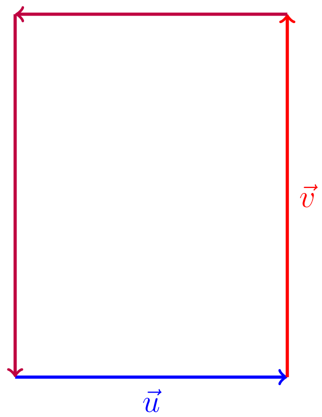
Lines are one-dimensional objects, and so vectors are also one-dimensional objects. Now what is the corresponding two-dimensional object? This is what a bivector is. At its core, it is an oriented plane segment wich three properties: the plane it is a segment of, its area (also called magnitude), and the orientation, which is the direction it points in (clockwise or anticlockwise on the plane). Just like how multiplying a vector by -1 changes the orientation, multiplying a bivector by -1 changes its orientation.

5.2 Using this to multiply perpendicular vectors

What we've just defined above is cool and all, but how does this help us with what we're trying to do? To help us, let's visualise this:



Here are two perpendicular vectors. Let's say that we want to calculate $\vec{u}\vec{v}$. There is a fairly natural bivector that we can draw here, shown below:



It just so happens that the magnitude of this bivector happens to be equal to the product of the magnitudes of the vectors. In other words, if we call the bivector B:

$$\|B\| = \|\vec{u}\|\,\|\vec{v}\|$$

The only thing that we have to sort out now is which orientation for the bivector we use. For this case, if we are finding $\vec{u}\vec{v}$, the product is the bivector that goes along $\vec{u}$ first and then along $\vec{v}$. By this definition, multiplying perpendicular vectors is anticommutative, meaning that if $\vec{u}$ and $\vec{v}$ are perpendicular, $\vec{v}\vec{u} = -\vec{u}\vec{v}$. Let's see where this can make our multiplication look more simple:

For 2D vectors:

$$\vec{u}\vec{v} = u_1v_1 + u_2v_2 + (u_1v_2 - u_2v_1)\hat{i}\hat{j}$$

For 3D vectors:

$$\begin{aligned}\vec{u}\vec{v} = &u_1v_1 + u_2v_2 + u_3v_3 + \\ &(u_1v_1 - u_2v_1)\hat{i}\hat{j} + \\ &(u_3v_1 - u_1v_3)\hat{k}\hat{i} + \\ &(u_2v_3 - u_3v_2)\hat{j}\hat{k}\end{aligned}$$

Just like we can add different basis vectors together, we can also add basis bivectors together to form one bivector. This leads to our product of two arbitrary vectors having a scalar part and a bivector part. The bivector part is called the outer product, and is written as $\vec{u} \wedge \vec{v}$, meaning that the product of two vectors is now:

$$\vec{u}\vec{v} = \vec{u} \cdot \vec{v} + \vec{u} \wedge \vec{v}$$

This now leads to the question of how we can add together a bivector and a scalar. We actually just leave it like this, similarly to how we add a real number and a complex number. This product is called the **geometric product**, and is the core part of geometric algebra.

6 The Geometric Interpretation

So far, we have been defining things and then simplifying what we got from a bracket expansion. Now, we will look at a more geometric interpretation.

Suppose we have two vectors $\vec{u}$ and $\vec{v}$, and we want to find $\vec{u}\vec{v}$ without knowing how they are expressed in basis vectors. We know how to multiply parallel and perpendicular vectors with only their magnitudes and directions, so we can do that. We can resolve $\vec{u}$ parallel to $\vec{v}$ and perpendicular to $\vec{v}$. We will call the component of $\vec{u}$ parallel to $\vec{v}$, $\vec{u}_{||}$, and the perpendicular component $\vec{u}_{\perp}$.

We know that the length of $\vec{u}_{||}$ is simply $\vec{v} \cdot \vec{u}$, so we just need to multiply that by the unit vector of $\vec{v}$ to find $\vec{u}_{||}$ and then multiply by $\vec{v}$:

$$\begin{aligned} \vec{u}_{||} &= \frac{\vec{v} \cdot \vec{u}}{||\vec{v}||^2}\vec{v} \\ \Rightarrow \quad \vec{u}_{||}\vec{v} &= \frac{\vec{v} \cdot \vec{u}}{||\vec{v}||^2}\vec{v}\vec{v} \\ &= \frac{\vec{v} \cdot \vec{u}}{||\vec{v}||^2}||\vec{v}||^2 \\ &= \vec{u} \cdot \vec{v} \\ &= ||\vec{u}||\vec{v}|| \cos\theta \end{aligned}$$

We have now found the scalar part of the geometric product in terms of just the magnitudes of the vectors and the angle between them. Now let's look at the bivector part. Keep in mind that the length of $\vec{u}$ perpendicular to $\vec{v}$ is $\|\vec{u}\| \sin\theta$.

$$\begin{aligned}
\vec{u}_\perp &= \vec{u} - \vec{u}_\parallel \\
&= \vec{u} - \frac{\vec{v} \cdot \vec{u}}{\|\vec{v}\|^2}\vec{v} \\
\Rightarrow \qquad \vec{u}_\perp \vec{v} &= \vec{u}\vec{v} - \vec{u} \cdot \vec{v} \\
&= \vec{u} \wedge \vec{v} \\
\Rightarrow \quad \|\vec{u}_\perp \vec{v}\| &= \|\vec{u}_\perp\| \|\vec{v}\| \\
&= \|\vec{u}\| \|\vec{v}\| \sin\theta
\end{aligned}$$

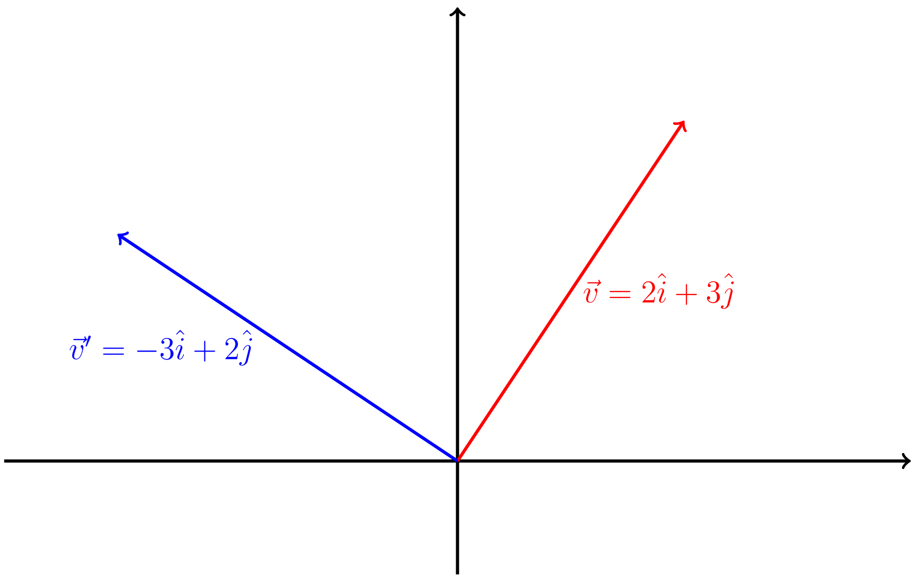
And now this is the bivector part of the geometric product in those same terms. Because it contains both $\cos\theta$ and $\sin\theta$, with the geometric product it is possible to determine the exact angle between them. For multiplying 2D vectors, we only have one unit bivector, so the geometric product of two 2D vectors is:

$$\vec{u}\vec{v} = \|\vec{u}\| \|\vec{v}\| (\cos\theta + \hat{\imath}\hat{\jmath} \sin\theta)$$

Huh, that's strange. That looks like how we express complex numbers in modulus-argument form. Let's see what happens when we square $\hat{\imath}\hat{\jmath}$.

$$\begin{aligned}
\left(\hat{\imath}\hat{\jmath}\right)^2 &= \hat{\imath}\hat{\jmath}\hat{\imath}\hat{\jmath} \\
&= -\hat{\imath}\hat{\imath}\hat{\jmath}\hat{\jmath} \\
&= -1
\end{aligned}$$

So does this mean that the unit bivector in two dimensions is i? As absurd as it sounds, it does. See what happens when we multiply a 2D vector by $\hat{\imath}\hat{\jmath}$:



That is a $\frac{\pi}{2}$ rotation, which is the same effect as multiplying by i on an Argand diagram. Part of the reason this happens is because in 2D, there is only one plane for a bivector to be a segment of, so the only thing that matters is the magnitude and orientation, meaning that we can generally treat bivectors in 2D as scalars. Hence, the product of two 2D vectors is a complex number.

7 Where is this useful?

It's cool that we can multiply vectors together, but there is no point in doing it if there isn't any use. Luckily, this is actually really useful. Because multiplying by the product of two vectors causes a rotational effect, this means that we can use the geometric product to represent rotations. In 2D, we can just do this using $e^{\hat{i}\hat{j}\theta}$ (remember that $\hat{i}\hat{j} = \mathrm{i}$). And then multiply the vectors that are being rotated by that, but in 3D it is a bit more complicated. Let's see what happens when we try to rotate a 3D vector $\frac{\pi}{2}$ about the z-axis. We will use $\vec{v} = x\hat{i} + y\hat{j} + z\hat{k}$ as our rotated vector. From using matrix multiplication to represent this rotation that we should get $x\hat{j} - y\hat{i} + z\hat{k}$. Because we are rotating in the xy plane, we will multiply by $\hat{i}\hat{j}$:

$$\begin{aligned}
\vec{v}\hat{i}\hat{j} &= (x\hat{i} + y\hat{j} + z\hat{k})\hat{i}\hat{j} \\
&= x\hat{i}\hat{i}\hat{j} + y\hat{j}\hat{i}\hat{j} + z\hat{k}\hat{i}\hat{j} \\
&= x\hat{i}\hat{i}\hat{j} - y\hat{j}\hat{j}\hat{i} + z\hat{i}\hat{j}\hat{k} \\
&= x\hat{j} - y\hat{i} + z\hat{i}\hat{j}\hat{k}
\end{aligned}$$

The x and y coefficients are correct, but our z turned into a multiple of $\hat{i}\hat{j}\hat{k}$. This is called a trivector, and just like how bivectors are oriented plane segments, a trivector is an oriented volume segment. We don't want that. But because the geometric product isn't commutative, let's try multiplying by the conjugate of $\hat{i}\hat{j}$, which is $-\hat{i}\hat{j} = \hat{j}\hat{i}$, and see if that changes anything:

$$\begin{aligned}
\hat{j}\hat{i}\vec{v} &= \hat{j}\hat{i}(x\hat{i} + y\hat{j} + z\hat{k}) \\
&= x\hat{j}\hat{i}\hat{i} + y\hat{j}\hat{i}\hat{j} + z\hat{j}\hat{i}\hat{k} \\
&= x\hat{j}\hat{i}\hat{i} - y\hat{j}\hat{j}\hat{i} - z\hat{i}\hat{j}\hat{k} \\
&= x\hat{j} - y\hat{i} - z\hat{i}\hat{j}\hat{k}
\end{aligned}$$

Almost the same thing happened again, however, this trivector is the opposite orientation. Maybe if I rotate by $\frac{\pi}{4}$ on the left and then by $\frac{\pi}{4}$ on the right, we will get rid of it. We first need to find out what we are going to be multiplying by, which we can do using Euler's formula:

$$\begin{aligned}
e^{\hat{i}\hat{j}\frac{\pi}{4}} &= \cos\frac{\pi}{4} + \hat{i}\hat{j}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{2}}{2} + \hat{i}\hat{j}\frac{\sqrt{2}}{2} \\
e^{-\hat{i}\hat{j}\frac{\pi}{4}} &= e^{\hat{j}\hat{i}\frac{\pi}{4}} \\
&= \cos\frac{\pi}{4} + \hat{j}\hat{i}\sin\frac{\pi}{4} \\
&= \frac{\sqrt{2}}{2} + \hat{j}\hat{i}\frac{\sqrt{2}}{2}
\end{aligned}$$

Now let's see if this works:

$$
\begin{aligned}
e^{-\hat{i}\hat{j}\frac{\pi}{4}}\vec{v}e^{\hat{i}\hat{j}\frac{\pi}{4}} &= \left(\frac{\sqrt{2}}{2}+\hat{j}\hat{i}\frac{\sqrt{2}}{2}\right)(x\hat{i}+y\hat{j}+z\hat{k})\left(\frac{\sqrt{2}}{2}+\hat{i}\hat{j}\frac{\sqrt{2}}{2}\right)\\
&= \left(\frac{\sqrt{2}}{2}x\hat{i}+\frac{\sqrt{2}}{2}x\hat{j}\hat{i}\hat{i}+\frac{\sqrt{2}}{2}y\hat{j}+\frac{\sqrt{2}}{2}y\hat{j}\hat{i}\hat{j}+\frac{\sqrt{2}}{2}z\hat{k}+\frac{\sqrt{2}}{2}z\hat{j}\hat{i}\hat{k}\right)\left(\frac{\sqrt{2}}{2}+\hat{i}\hat{j}\frac{\sqrt{2}}{2}\right)\\
&= \left(\frac{\sqrt{2}}{2}x\hat{i}-\frac{\sqrt{2}}{2}y\hat{i}+\frac{\sqrt{2}}{2}y\hat{j}+\frac{\sqrt{2}}{2}x\hat{j}+\frac{\sqrt{2}}{2}z\hat{k}-\frac{\sqrt{2}}{2}z\hat{i}\hat{j}\hat{k}\right)\left(\frac{\sqrt{2}}{2}+\hat{i}\hat{j}\frac{\sqrt{2}}{2}\right)\\
&= \frac{1}{2}x\hat{i}+\frac{1}{2}x\hat{i}\hat{i}\hat{j}-\frac{1}{2}y\hat{i}-\frac{1}{2}y\hat{i}\hat{i}\hat{j}+\frac{1}{2}y\hat{j}+\frac{1}{2}y\hat{j}\hat{i}\hat{j}+\frac{1}{2}x\hat{j}+\frac{1}{2}x\hat{j}\hat{i}\hat{j}+\frac{1}{2}z\hat{k}+\frac{1}{2}z\hat{k}\hat{i}\hat{j}\\
&+-\frac{1}{2}z\hat{i}\hat{j}\hat{k}-\frac{1}{2}z\hat{i}\hat{j}\hat{k}\hat{i}\hat{j}\\
&= \frac{1}{2}x\hat{i}+\frac{1}{2}x\hat{j}-\frac{1}{2}y\hat{i}-\frac{1}{2}y\hat{j}+\frac{1}{2}y\hat{j}-\frac{1}{2}y\hat{i}+\frac{1}{2}x\hat{j}-\frac{1}{2}x\hat{i}+\frac{1}{2}z\hat{k}+\frac{1}{2}z\hat{i}\hat{j}\hat{k}\\
&-\frac{1}{2}z\hat{i}\hat{j}\hat{k}+\frac{1}{2}z\hat{k}\\
&= x\hat{j}-y\hat{i}+z\hat{k}
\end{aligned}
$$

This works! We can generalise this by saying that a rotation in the plane of the unit bivector $\hat{I}$ by θ on a vector $\vec{v}$ is given by:

$$e^{-\hat{I}\frac{\theta}{2}}\vec{v}e^{\hat{I}\frac{\theta}{2}}$$

The value $e^{\hat{I}\frac{\theta}{2}}$ is called a rotor when used in this way, and it is in the form: $a+b\hat{i}\hat{j}+c\hat{j}\hat{k}+d\hat{k}\hat{i}$, so this means that we can represent a 3D rotation with just four values. This is advantageous to doing rotations with a matrix when doing computer graphics because you only store four values instead of nine, saving memory.

8 Conclusion

You have now reached the end of this very basic introduction and explaination of geometric algebra, as well as gone through one of the uses of it.

There are so many other uses and fascinating qualities about it that I would have gone through if there wasn't a word limit to this essay. If you're interested, look at what using this can do to Maxwell's Equations, and how the above rotations link to the quaternions. Thank you for reading this essay and I hope that you have had as good a time doing so as I had making it.