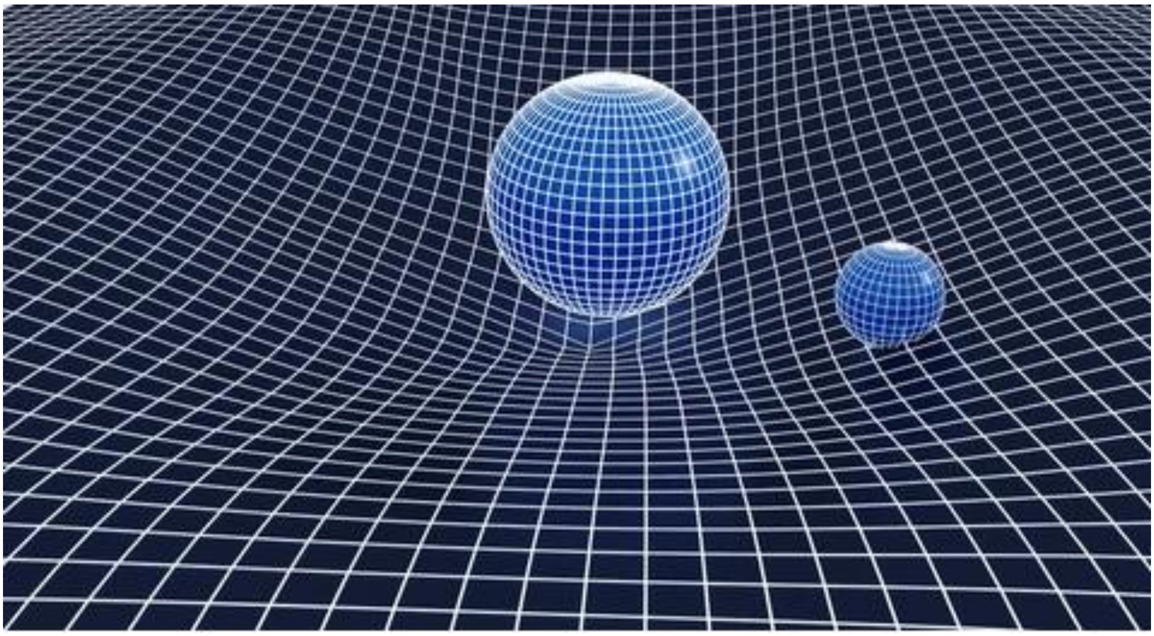
Why curved spaces hate coordinates

The mathematical limits of general relativity



Aureliano Faraldos Wiegand

Curved spaces, known mathematically as manifolds, cannot be faithfully mapped by a coordinate system. They resist the precise, distortion-free coordinate descriptions we take for granted in Euclidean geometry. General relativity describes gravity not as a force but as the curvature of spacetime: a 4-dimensional curved manifold. It is precisely this curved geometry that necessitates the mathematical framework developed in this essay.

1. What is a Coordinate System?

A coordinate system on a space gives each geometric entity, such as points, lines, or surfaces, a unique label made of some set of numbers. The number of coordinates required depends not just on the dimension of the space but on the nature of the entity being labelled. In 3-dimensional space a point requires 3 values, but a line requires more, reflecting its additional degrees of freedom. The number of coordinates used to describe the same entity can also vary between coordinate systems: Plücker coordinates describe a line in 3D space using 6 values, but only 4 are required.

For a coordinate system to be valid it must satisfy several key requirements. The mapping must be one-to-one: every entity needs a unique label, and each label must correspond to exactly one entity. When this breaks down it is called the redundancy problem. Closely related is invertibility: given any coordinate label it must be possible to uniquely recover the entity it describes; if one fails, the other follows. A coordinate system must also preserve distances: nearby entities must receive nearby coordinate values and distances must be recoverable from the labels themselves. Finally, a coordinate system must cover the entire space with no region left unaddressed. These are not the only requirements a coordinate system must satisfy, but they are the most relevant to the argument developed here.

2. Intrinsic and Extrinsic Geometry

The geometry of a manifold can be understood in two fundamentally different ways. Extrinsic geometry describes a space from the outside: in terms of how it sits within a higher dimensional ambient space. Intrinsic geometry describes a space purely from within, using only measurements made inside the space itself, with no reference to any surrounding space.

The surface of a sphere makes this concrete. Extrinsically, the surface sits inside a 3-dimensional space and its curvature is obvious when viewed from outside. Intrinsically, the curvature is equally detectable from within: triangles drawn on the surface have angles summing to more than 180 degrees, and geodesics, the straightest possible paths on the surface, that begin parallel will eventually meet. These are purely intrinsic measurements. The curvature is therefore not a property of how the sphere sits in 3-dimensional space; it is a fundamental property of the surface itself.

This known as Gauss's Theorema Egregium, Latin for "remarkable theorem." No matter how a curved surface is described, whether by bending it, relabelling it, or choosing a different coordinate system, the curvature cannot be made to disappear. The curvature of a manifold is an intrinsic invariant: it belongs to the geometry of the space itself and is independent of any coordinate system or ambient space.

3. Why Coordinate Systems Fail on Curved Manifolds

Since curvature is intrinsic and cannot be eliminated, any coordinate system placed on a curved manifold must contend with it honestly. No coordinate choice can flatten a curved space.

The most fundamental problem is distortion. In flat space it is always possible to find a coordinate system where the distance between nearby points satisfies:

$$ds^2 = dx^2 + dy^2 + ...$$

This is Pythagoras's theorem generalised to any number of dimensions, and the fact that it holds everywhere simultaneously is definitionally what it means for a space to be flat. On a curved manifold the curvature stretches and bends the geometry in ways this formula cannot capture globally: in some regions, distances are compressed, in others stretched. Any coordinate system on a curved manifold will always distort distances somewhere, meaning no faithful coordinate description exists.

A natural first attempt to sidestep this is to borrow coordinates from a higher dimensional ambient space. The surface of a sphere sits inside 3-dimensional space, so one might assign each point its (x, y, z) coordinates from the surrounding space. This appears to work as a labelling scheme, but it is fundamentally extrinsic: the coordinates belong to the 3-dimensional ambient space, not to the 2-dimensional surface. Any attempt to measure distances or curvature on the surface using these extrinsic coordinates fails immediately, because the coordinates carry no intrinsic geometric information. The redundancy problem here is particularly severe: the extrinsic system contains labels for infinitely many entities that do not exist on the manifold at all.

The global coverage problem presents a further obstruction. At the north and south poles of the Earth every longitude line meets, meaning infinitely many coordinate labels describe the same point, directly violating the one-to-one requirement. These points become undefined, producing two distinct failures simultaneously: the redundancy problem and complete breakdown of coverage. It is a general theorem that any attempt to describe a curved manifold with a single global coordinate system will produce distortion, redundancy, loss of invertibility, and breakdown.

4. The Consequences for General Relativity

General relativity describes gravity as the curvature of spacetime: a 4-dimensional curved manifold. Every failure established in Section 3 therefore applies directly to the mathematics of the theory.

In Newtonian physics space is flat, existing as a fixed background with a natural coordinate system all observers agree on. Because the underlying space is flat, the coordinates faithfully reflect the geometry everywhere; there is no distortion, no breakdown, no redundancy. General relativity dismantles this entirely. By treating gravity as curvature rather than force it removes the flat background space and with it the natural coordinate system Newtonian physics relied upon. Any quantity expressed in one coordinate system will have entirely different numerical values in another, even though the underlying physical reality is identical. A theory built on coordinate dependent quantities would give different predictions depending on an arbitrary choice that does not correspond to any physical feature of spacetime: which is physically meaningless.

There is a partial escape: working locally. Consider standing on the surface of the Earth. Despite being a curved manifold, the ground appears perfectly flat because you are so small relative to the Earth that the curvature is negligible at your scale. Zoom in close enough to any point on a curved manifold and the same thing happens; the geometry becomes approximately flat, and a coordinate system can be assigned that satisfies all the requirements of Section 1 to a good approximation. In general relativity this is the equivalence principle: in any sufficiently small region of spacetime the physics looks identical to flat spacetime. The difficulty is that gravity is not local; it is a property of how spacetime curves across large regions. The moment a calculation reaches beyond one local region into another the coordinate systems of the two patches disagree. Local coordinates are a useful approximation but cannot serve as the foundation for a complete theory of gravity.

The crisis becomes sharpest when taking derivatives. In flat space taking a derivative is straightforward because the geometry is the same everywhere. On a curved manifold the geometry varies from point to point, and any derivative involves comparing a quantity at one point with its value at a nearby point. Because the geometry has changed between those points the comparison inevitably mixes two contributions together. Consider a velocity V^μ on a curved manifold. The ordinary derivative is:

$$\frac{\partial V^\mu}{\partial x^\nu}$$

Under an arbitrary coordinate change of $x^\mu \to x^{\mu'}$ this transforms as:

$$\frac{\partial V^{\mu'}}{\partial x^{\nu'}} = \frac{\partial x^{\mu'}}{\partial x^\mu}\frac{\partial x^\nu}{\partial x^{\nu'}}\frac{\partial V^\mu}{\partial x^\nu} + V^\mu \frac{\partial^2 x^{\mu'}}{\partial x^{\nu'}\partial x^\mu}$$

The first term is the genuine physical change in velocity. The second term is generated entirely by the coordinate system twisting across the curved manifold and carries no physical information whatsoever. The ordinary derivative cannot separate these two contributions, producing results that depend on the arbitrary choice of coordinates rather than on the actual physics. General relativity therefore cannot be formulated using coordinate dependent mathematics.

5. Tensors: The Solution

The solution is a mathematical framework in which the fundamental objects and equations are guaranteed to be coordinate independent. That framework is the theory of tensors.

To understand tensors, it is necessary to first understand vectors. A vector is a quantity with both magnitude and direction: velocity, acceleration, and force are all examples. A vector V can be expressed in terms of its components along the coordinate axes:

$$V = V^x\hat{x} + V^y\hat{y} + V^z\hat{z}$$

The vector itself is a real physical object. Its components however are coordinate dependent: rotate the coordinate axes and the components change, even though the underlying vector has not moved. On a curved manifold this becomes severe; the coordinate axes tilt and stretch across the manifold, meaning any equation written directly in terms of vector components will be coordinate dependent: valid in one system but not in another.

A tensor is a generalisation of a vector designed to solve precisely this problem. A tensor can have any number of indices and transforms in a controlled, consistent way under coordinate changes. The key property: if a tensor equation holds in one coordinate system, it holds in all coordinate systems, and so, choice of coordinate system becomes redundant. Any coordinate system you choose will break dwon but since tensors are coordinate independent that does not matter. You can still do mathematics with them even if the coordinate system breaks down.

A tensor with no indices is a scalar. A tensor with one index is a vector. A tensor with two indices $T^{\mu\nu}$ gives a 4x4 grid of 16 numbers in 4-dimensional spacetime. Under a coordinate change it transforms as:

$$T^{\mu'\nu'} = \frac{\partial x^{\mu'}}{\partial x^{\mu}} \frac{\partial x^{\nu'}}{\partial x^{\nu}} T^{\mu\nu}$$

Unlike the ordinary derivative there is no arbitrary second term. The tensor equation $T^{\mu\nu} = S^{\mu\nu}$ is either true in all coordinate systems or false in all of them: exactly the property needed for physically meaningful equations on a curved manifold.

Tensors give us the ability to overcome the curvature of space time in the following ways:

The Metric Tensor. The most important tensor in general relativity is the metric tensor $g_{\mu\nu}$: a 4x4 grid of 16 numbers at each point in spacetime encoding distances, angles, the rate at which time flows, and the paths that light follows. Instead of the flat formula:

$$ds^2 = dx^2 + dy^2 + ...$$

the general distance formula on a curved manifold is:

$$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$

where repeated indices indicate a sum over all four dimensions, known as Einstein summation convention. In flat spacetime $g_{\mu\nu}$ takes the simple diagonal form:

$$g_{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

The -1 in the first position reflects the special role of time: time and space contribute differently to distances in spacetime. In curved spacetime the components vary across the manifold encoding the curvature at each point. Because $g_{\mu\nu}$ is a tensor the underlying geometry it encodes remains the same regardless of coordinates: it is the coordinate independent description of spacetime geometry that coordinate systems alone could never provide.

The Covariant Derivative. The ordinary derivative:

$$\frac{\partial V^\mu}{\partial x^\nu}$$

is not a tensor; it picks up that same arbitrary second term under coordinate changes. The covariant derivative $\nabla_\nu V^\mu$ adds a correction term that exactly cancels this term:

$$\nabla_\nu V^\mu = \frac{\partial V^\mu}{\partial x^\nu} + \Gamma^\mu_{\nu\lambda} V^\lambda$$

The Christoffel symbols $\Gamma^\mu_{\nu\lambda}$, calculated directly from the metric tensor, encode how the coordinate system twists and stretches across the manifold. By subtracting out this coordinate twisting the covariant derivative isolates genuine physical change, leaving a true tensor. Every ordinary derivative in physics must be replaced by the covariant derivative on a curved manifold.

The Einstein Field Equations. With these tools the central equation of general relativity can be written:

$$G_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

Every object is a tensor: the equation is coordinate independent and holds across the entire curved manifold simultaneously. $T_{\mu\nu}$ is the stress-energy tensor: a 4x4 tensor encoding the distribution of matter and energy throughout spacetime. Its components describe energy density, momentum density, pressure, and stress at each point; the T^{00} component encodes how much energy per unit volume is present there. The off-diagonal components encode momentum flow and pressure. The stress-energy tensor is the complete mathematical description of everything that curves spacetime. Every form of matter, energy, pressure, and momentum contributes to $T_{\mu\nu}$.

$G_{\mu\nu}$ is the Einstein tensor. It is also a 4×4 tensor and encodes the curvature of spacetime at each point. It is constructed from the metric tensor, and its covariant derivatives can be written as:

$$G_{\mu\nu} = R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

$R_{\mu\nu}$ is the Ricci tensor, a measure of how volumes are distorted by curvature, and R is the Ricci scalar, a single number summarising the overall curvature at a point. Both are constructed from the Riemann curvature tensor: the complete description of curvature requiring 20 independent numbers in 4-dimensional spacetime.

$\frac{8\pi G}{c^4}$ is the coupling constant between matter and geometry. The presence of c^4 in the denominator makes this an extraordinarily small number: which is why enormous amounts of matter and energy are needed to produce detectable curvature or gravity.

The equation says: the curvature of spacetime at each point, encoded in $G_{\mu\nu}$, is determined by the distribution of matter and energy there, encoded in $T_{\mu\nu}$ Mass and energy curve spacetime; curved spacetime determines how mass and energy move. Because both sides are tensors this relationship holds in every coordinate system simultaneously: it is a statement about the geometry of spacetime itself, entirely independent of any arbitrary coordinate choice. The coordinate independence of the Einstein field equations is not merely a mathematical convenience; it is a physical necessity forced upon the theory by the intrinsic curvature of spacetime.

6. Conclusion

Curved manifolds resist faithful coordinate descriptions because their curvature is intrinsic: it cannot be removed by any relabelling, embedding, or coordinate choice. General relativity, built on a 4-dimensional curved manifold, therefore cannot be formulated in coordinate dependent mathematics without producing physically meaningless results. Tensors resolve this by providing mathematical objects whose equations are coordinate independent by construction: the choice of coordinate system becomes redundant. The metric tensor encodes the geometry of spacetime, the covariant derivative replaces the ordinary derivative, and the Einstein field equations express the relationship between curvature and matter in a form that holds across the entire manifold simultaneously.