Why do we pi?

By Tom Tregenza

1. Introduction

Throughout the years there have been so many different ways to approximate the value of pi, with some more accurate than others. Today I will be exploring some of my favourite and the most influential methods of calculating the most beautiful constant in mathematics. This will then lead me to ask the question of "why" it is so important.

2. A "Brief" History Lesson

The first known records of pi being used was in ancient Babylon in about 2000BC where clay tablets have been found using π =3 to calculate the area of circles using the formula we commonly know today of Area = π x radius^2. You could say the Babylonians were the πoneers of pi!

Now, let us fast forward about 159π years to ancient Egypt where a scribe by the name of Ahems was also doing some calculations. His papyrus (ancient writing material) has been found to contain some early calculations and maths problems using the newly made revolutions in numerals and fractions which then allowed him to calculate pi to be around $\frac{256}{81}$ or 3.16. Yet, this approximation is only correct to 1 significant figure when compared to the true value of pi but nevertheless a great leap in the history of mathematics.

Unfortunately, there are many places in ancient history where pi has been attempted to be calculated so I won't be able to name them all. So, to try and keep under the word limit, the main places of origin of pi were: Babylon, Greece, Egypt, China and India. One of the best approximations of pi to come out of ancient times was Archimedes' polygon approximation in roughly 250 BC. In his approximation, he didn't come up with an attempt at the true value of pi, rather he formulated a tight range that pi falls in between. Archimedes' approximation was purely geometric, as advanced calculus and other areas of mathematics hadn't been discovered yet. The main idea to his approximation was using 2 polygons, one slightly smaller and one slightly larger than the same circle. This then allowed him to calculate the perimeter of each polygon and then come up with a value of pi from each polygon. This meant that the true value of pi was between the 2 values he got. Using this method, Archimedes found that pi was between $\frac{223}{71}$ and $\frac{22}{7}$, which agree to 3 significant figures, therefore Archimedes calculated pi to be 3.14 to 3 significant figures.

3.1 My Favourite Approximations of Pi

In this section I will be going through some my favourite approximations of pi to better understand their main techniques in hopes to really understand "why". Before I start I would like to note that many of the approximations I am about to state and some of the ones I have already stated are likely to have been discovered by multiple people across the globe, however I will only state the person that is most commonly linked to each approximation from a reliable source.

3.2 The Newton Approximation

In about 1660, Sir Isaac Newton used his and Gottfried Wilhelm Leibniz's newly discovered techniques of calculus to come up with his own approximation of pi. The invention of calculus allowed mathematicians to perform calculations that were once not possible. The main techniques used in this approximation were integration and the infinite series of the binomial expansion (which was also discovered by Newton, isn't he clever!).

His approximation went like this:

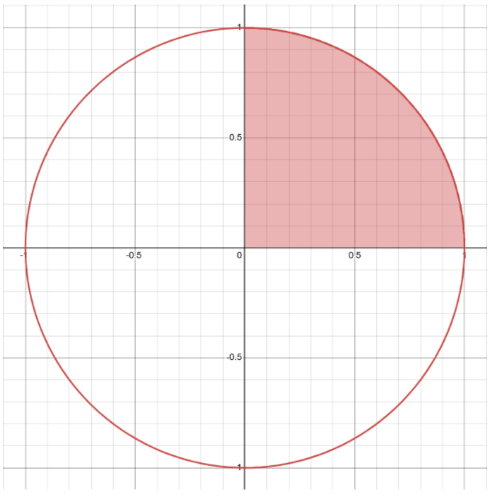
Say you have a circle of radius 1, given by the equation $x^2 + y^2 = 1$

By this point in history it was quite well known the area of a circle can be given by this formula here: $A = \pi r^2$

∴ The area of this circle would be $\pi * 1^2 = \pi$

Now if we were to integrate the whole circle, this process would be a lot more complex due to the equation not being in explicit form.

Instead we rearrange the equation to get $y = \sqrt{1 - x^2}$ taking the positive square root because we will only be integrating the top right hand quarter of the circle.



We have already calculated the area of this circle to be π $\therefore$ the area of this quarter will be $\frac{\pi}{4}$.

Now when we integrate the rearranged equation from before between the bounds of 0 and 1, it will be equal to $\frac{\pi}{4}$.

$$\int_0^1 \sqrt{1-x^2}\, dx = \frac{\pi}{4}$$

Now our approximation comes from multiplying the integral by 4 to get.

$$4\int_0^1 \sqrt{1-x^2}\, dx = \pi$$

Next we have to use the binomial expansion to expand $\sqrt{1-x^2}$ into an infinite series, however we will only expand up to 22 terms as Newton did.

Here are all 22 terms:

$$1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-\frac{7x^{10}}{256}-\frac{21x^{12}}{1024}-\frac{33x^{14}}{2048}-\frac{429x^{16}}{32768}-\frac{715x^{18}}{65536}-\frac{2431x^{20}}{262144}$$
$$-\frac{4199x^{22}}{524288}-\frac{29393x^{24}}{4194304}-\frac{52003x^{26}}{8388608}-\frac{185725x^{28}}{33554432}-\frac{334305x^{30}}{67108864}-\frac{9694845x^{32}}{2147483648}$$
$$-\frac{17678835x^{34}}{4294967296}-\frac{64822395x^{36}}{17179869184}-\frac{119409675x^{38}}{34359738368}-\frac{883631595x^{40}}{274877906944}$$
$$-\frac{1641030105x^{42}}{549755813888}$$

It is very tedious trying to type in all 22 terms over and over again in word so save time and space will only be writing the first 6 terms, but do trust that I am doing calculations to all the terms.

Now,

$$4\int_0^1 1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}-\frac{5x^8}{128}-\frac{7x^{10}}{256}\dots\, dx = \pi$$

$$\therefore 4\left[x-\frac{x^3}{6}-\frac{x^5}{40}-\frac{x^7}{112}-\frac{5x^9}{1152}-\frac{7x^{11}}{2816}\dots\right]_0^1 = \pi$$

The lower bound being equal to 0 will eliminate all the terms in the subtracted bracket.

$$\therefore 4\left(1-\frac{1}{6}-\frac{1}{40}-\frac{1}{112}-\frac{5}{1152}-\frac{7}{2816}\dots\right)-(0) = \pi$$

Multiplying through by 4:

$$\pi = 4 - \frac{4}{6} - \frac{4}{40} - \frac{1}{112} - \frac{5}{1152} - \frac{7}{2816} \cdots$$

This led Newton to calculate pi to 16 decimal places:

$$\pi \approx 3.1415926535897932$$

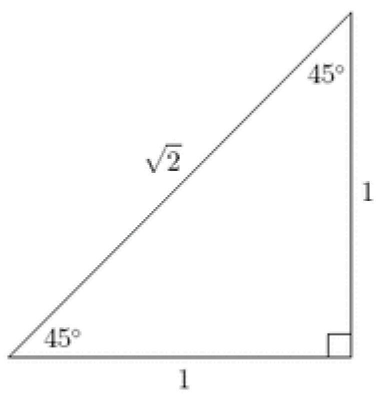
Overall a very elegant approximation incorporating some of Newton's newly discovered techniques. Some of you may be thinking "why don't you just use a substitution for the integral to get the exact answer?". The reason we can't do that is because we already know the integral is exactly equal to pi from our geometric proof. This means we would just be proving that the area of the circle is pi, which we already know.

3.3 The Madhava-Leibniz Approximation

The Madhava-Leibniz Approximation was first discovered in India by Madhava of Sangamagrama in the 14th century (allegedly in his sleep). It was then independently rediscovered by James Gregory in 1671 and published by Gottfried Wilhelm Leibniz in 1676 which led to the series being named after him. The main principles in this approximation are basic trigonometry and the Taylor series expansion of arctangent.

The approximation was this:

Suppose you have an isosceles triangle with its equal sides of length 1,



Now we know that the tangent function of the 45 degrees in this triangle is equal to the opposite over the adjacent:

$$\tan(45) = \frac{\text{opp}}{\text{adj}}$$

$$\therefore \tan(45) = \frac{1}{1} = 1$$

However, for the Taylor series of arctangent to work we will need to be working in radians.

Converting to radians we get:

$$\tan\left(\frac{\pi}{4}\right) = 1$$

$$\therefore \frac{\pi}{4} = \arctan(1)$$

Next, we will use the Taylor series expansion of arctangent to get an infinite sum to calculate pi.

The Taylor series of arctangent is given by:

$$\arctan(x) = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} \ldots$$

Or as an infinite sum:

$$\sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{2k+1}$$

Now when we substitute 1 into the series we get:

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} \ldots$$

$$\therefore \pi = 4 - \frac{4}{3} + \frac{4}{5} - \frac{4}{7} \ldots$$

Now, despite the elegance and genius behind this approximation, it does converge very slowly requiring 5 billion terms to calculate only 10 digits of pi. I discovered the unhurried nature of this approximation when I coded it in python to see how many correct digits I could get out.

```
pi = 0
for i in range(100):
    pi=pi+4*((-1)**i)/(2*i+1)
print(pi)
```

When I used 100 terms I received pi to be 3.1315929035585537 which is only correct to 2 significant figures. As you can see it would be very time consuming and unrewarding to manually calculate pi with this approximation which is why it is rarely used in systems that require high speed calculations of pi.

For reference, when I used 1 million terms I got pi correct to 7 significant figures.

I then attempted 1 billion terms… Let's just say I aborted that before my laptops fans caused it to fly out my window.

3.4 Machin Like Formulae

We now know that the Madhava-Leibniz approximation converges very slowly and is subsequently a bad formula to use for calculating pi. Enter Machin like formulae: Machin like formulae take the genius and elegance behind the Madhava-Leibniz approximation and make it much more efficient allowing people to calculate hundreds of digits of pi so much easier. The name comes from when John Machin managed to calculate pi to 100 decimal places which was a record at the time. He did this using the unexplainable formula at the time:

$$\pi = \left(\frac{16}{5} - \frac{4}{239}\right) - \frac{1}{3}\left(\frac{16}{5^3} - \frac{4}{239^3}\right) + \frac{1}{5}\left(\frac{16}{5^5} - \frac{4}{239^5}\right) \dots$$

This was then later discovered to be an expansion of the formula:

$$\frac{\pi}{4} = 4 \arctan\left(\frac{1}{5}\right) - \arctan\left(\frac{1}{239}\right)$$

This formula is also just an expanded version of the Madhava-Leibniz formula, however due to there being multiple arctangent terms, when you expand using the Taylor series it will converge much quicker. This then led to many more expanded versions of the formula each with more arctangent terms and increasing convergence speeds. We call this family of formulae Machin like formulae, and they are commonly used in modern day computations of pi.

The way we derive the formula above is using the addition formula for arctangent.

The derivation goes like this:

First, we begin with,

$$2 \arctan \frac{1}{5} = \arctan \frac{1}{5} + \arctan \frac{1}{5}$$

Next, we need to use the addition formula of arctangent as seen below:

$$\arctan \frac{a_1}{b_1} + \arctan \frac{a_2}{b_2} = \arctan \frac{a_1 b_2 + a_2 b_1}{b_1 b_2 - a_1 a_2}$$

Now, using the formula:

$$2 \arctan \frac{1}{5} = \arctan \frac{1 * 5 + 1 * 5}{5 * 5 - 1 * 1}$$

$$2 \arctan \frac{1}{5} = \arctan \frac{10}{24}$$

$$2 \arctan \frac{1}{5} = \arctan \frac{5}{12}$$

Therefore, we can now also say:

$$4 \arctan \frac{1}{5} = 2 \arctan \frac{1}{5} + 2 \arctan \frac{1}{5}$$

$$4 \arctan \frac{1}{5} = \arctan \frac{5}{12} + \arctan \frac{5}{12}$$

Using the addition formula again,

$$4 \arctan \frac{1}{5} = \arctan \frac{5 * 12 + 5 * 12}{12 * 12 - 5 * 5}$$

$$4 \arctan \frac{1}{5} = \arctan \frac{120}{119}$$

Next, we start with quite an obvious yet necessary line of working:

$$4\arctan\frac{1}{5} = 4\arctan\frac{1}{5}$$

Now, remembering that arctangent 1 (or arctangent $\frac{1}{1}$) = $\frac{\pi}{4}$, which means we can take both away from either side:

$$4\arctan\frac{1}{5} - \frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{1}$$

$$4\arctan\frac{1}{5} - \frac{\pi}{4} = \arctan\frac{120}{119} + \arctan\frac{-1}{1}$$

Using the addition formula one last time:

$$4\arctan\frac{1}{5} - \frac{\pi}{4} = \arctan\frac{120 * 1 + 119 * -1}{119 * 1 - 120 * -1}$$

$$4\arctan\frac{1}{5} - \frac{\pi}{4} = \arctan\frac{1}{239}$$

Finally, we just rearrange to get the formula we had before:

$$\frac{\pi}{4} = 4\arctan\frac{1}{5} - \arctan\frac{1}{239}$$

Hopefully you can see how repetitive this process is, which is what allows people to just keep repeating the steps over and over again until they come up with a formula with many more arctangent terms and faster converging speeds. Here are some examples:

$$\frac{\pi}{4} = \arctan\frac{1}{2} + \arctan\frac{1}{3},$$

$$\frac{\pi}{4} = 12\arctan\frac{1}{49} + 32\arctan\frac{1}{57} - 5\arctan\frac{1}{239} + 12\arctan\frac{1}{110443}$$

Or my personal favourite:

$$\frac{\pi}{4} = 22\arctan\frac{1}{28} + \arctan\frac{17445074821803283266854565127}{98646395734210062276153190241239}$$

Overall, Machin like formulae are an amazing continuation of the innovative Madhava – Leibniz approximation and are still used today in modern systems.

4. But Why?

I'm sure many of you are thinking "Why?", "What's the point?", "Who cares?". To that I say… there isn't necessarily a major point to calculating pi to thousands or millions of decimal places. In light of the current Artemis II mission, a good example is that NASA revealed that they only use 15 decimal places of pi when performing vital calculations in order to get our astronauts safely to the moon. This just shows that even at the heights of human engineering and technology, only less than 20 digits of pi is used. Another good example is large scale bridges. The mathematics behind them only use less than 10 digits of pi to keep your car from swimming with the fishes.

Now, despite my examples backing up why it is a complete waste of time to calculate pi to billions of decimal places, there is a real reason to what drives people to perform these crazy calculations. That reason: Human Nature. Since the dawn of time, humans have been completing crazy tasks solely for the purpose of doing them with no real benefit other than the thrill of completing them. The indomitable human spirit is sometimes so strong, it pushes people to their limits causing them to defy what people once thought wasn't humanly possible. There are plenty examples: Sky diving, running an ultra-marathon, climbing a mountain, sailing the Pacific Ocean, solving a Rubix cube, calculating 1 trillion digits of pi…

Calculating pi has been one of the most common and competitive tasks since the dawn of mathematics. It is more than just discovering new digits of an irrational number. I think that approximating pi is like the fashion show of mathematics. Will anyone actually use pi to this many significant figures? No. Is it a great way to showcase new and innovative techniques in mathematics? Yes. As many of you would have seen, there was a multitude of techniques used for the approximations I went through touching many areas of pure mathematics. For this reason, next time you see your friend sat at their desk attempting to calculate as many digits of pi as they can, don't judge.

Just enjoy the fashion show.