Why Infinity Broke Mathematics – And How We Tamed It

By: Aditya Mehta

1 Introduction

To begin this essay, I would like for you to visualise an idea.

Imagine a hotel with infinitely many rooms, with every room being occupied. A new guest arrives one day, late at night, asking for a room to stay in. If this were an ordinary hotel, this would be impossible; but in this hotel, the manager calmly asks the guest in Room 1 to move to Room 2, the guest in Room 2 to move to Room 3 and so on. When every guest shifts forward one room, Room 1 becomes free for the new guest.

Despite being "full", the hotel has space.

This thought experiment, known as Hilbert's paradox of the Grand Hotel, was proposed by mathematician David Hilbert in the 1920s and exposes something unsettling at the time: infinity does not obey the arithmetic of "ordinary" numbers [1]. For a finite number n, we know that $n+1 > n$; however, with infinity, adding one does not seem to change anything at all.

Infinity seems harmless, yet when mathematicians attempted to treat it, it destabilised the foundations of their subject, creating paradoxes and logical contradictions. Mathematics, the subject based on certainty, was suddenly unsure of itself.

This essay will explore how infinity nearly broke mathematics and how through the invention of limits, set theory and axiomatic systems, mathematicians learned how to control it.

2 The Paradox of Infinite Processes

Before the formal set theory was established, infinity was already troubling mathematicians.

Consider Zeno's paradox [2]: to walk across a room, you must first travel half the distance. Then half the half the remaining distance, then half again and so on. This produces an infinite sequence of steps:

$$\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \ldots$$

This creates the question, if infinitely steps are required, then how will the distance ever be met? The solution comes from recognising that an infinite number of terms can sum to a finite value. The geometric series

$$\sum_{n=1}^{\infty} \frac{1}{2^n}$$

Converges and by using the formula for a geometric series

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$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}, \qquad |r| < 1,$$

From this, we find that

$$\sum_{n=1}^{\infty} \frac{1}{2^n} = 1.$$

An infinite process can provide a finite answer.

However, this solution required precision. Early calculus, developed by booth Newton and Leibniz, relied on “infinitesimals” which is a quantity so infinitely small but not zero. Although the method worked well, its logical foundation was shaky. Dividing by infinitesimals, cancelling them and treating both as zero and not zero was “dangerous” mathematically.

This eventually led to the concept of limits:

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Instead of using infinitesimals, the derivative describes what happens as *h* approaches zero. The idea of infinitely smaller values was no longer considered as objects but more as a process. This was the first major attempt to control infinitely.

3 Counting the Infinite

For centuries, infinity was viewed as a single concept but the changed dramatically with the work of Georg Cantor in the late nineteenth century [3].

Cantor introduced the idea that the size of a set (even an infinite one) can be defined through one-to-one correspondence. If two sets can be paired element for element without any leftovers, they have the same cardinality – the size of a set.

First, look at the natural numbers

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}.$$

No consider the even numbers:

$$E = \{2, 4, 6, 8, \dots\}$$

You might think that E is half as large but the function $f(n) = 2n$ pairs each natural number with exactly one even number meaning that no element is unmatched. Therefore, both sets have the same cardinality

Infinity defies human intuition as it shows how a whole can equal one of its parts. Cantor called these types of set countably infinite and denoted their size by $\aleph_0$ (aleph-null).

4 The Diagonal Argument

Cantor asked whether the real numbers between 0 and 1 are countable. If we considered that this is true, then we could list them like:

1. 0.123456…
2. 0.988765…
3. 0.314159…
4. 0.500000…
5. …

Now, construct a new number by altering the first digit of the first number, and so on. For example, if the nth digit of the nth number is 5 then replace it with 6, otherwise replace it with 5. This means that the resulting number is different from every entry on the list in at least one decimal place. This means that it cannot appear on the list.

No list can contain all real numbers.

Thus, the set of real numbers is uncountable. The size of the set (cardinality) is larger than $\aleph_0$. Cantor denoted this size by $2^{\aleph_0}$, also known as the cardinality of the continuum.

Infinity comes in different sizes.

This discovery destroyed centuries of assumptions. Mathematics no longer contained a single infinity, but a hierarchy of them.

5 The Crisis of Foundations

Cantor's set theory treated sets as collections defined by properties; however, this approach led to contradictions.

Imagine there was "the set of all sets that do not contain themselves."

If it contains itself then that should mean that it also doesn't contain itself. If it does not contain itself, then it should contain itself.

This is the Bertrand Russell's paradox.

This contradiction was devastating and led to mathematicians to respond by attempting to rebuild their subject from strict axioms (this is a statement accepted as true without proof and act as a foundational building block for establishing further truths).

One of the most influential systems was Zermelo-Fraenkel set theory with the Axiom of Choice; this essentially stated that you can only build sets using very fixed rules and you cannot just make up any set that you want. The Axiom of Choice is an extra rule that lets you pick elements from many sets even if there is no explicit formula for choosing them.

The lesson was clear – infinity cannot be handled informally.

6 The Arithmetic of the Infinite

Cantor's theory revealed a strange phenomenon. For infinite cardinals:

$$\aleph_0 + \aleph_0 = \aleph_0$$

$$\aleph_0 \times \aleph_0 = \aleph_0$$

But also more complicating and surprisingly

$$|R| = 2^{\aleph_0}$$

And…

$$p|R| = 2^{2^{\aleph_0}}$$

The power set of R, written a p modulus R, is the set of all possible subsets of R, where R represents the real numbers. Essentially, this means that there is no “largest” infinity. Infinity creates and endless tower of ever-larger infinities.

The very concept of size was changed.

7 Infinity and Logic

A new problem arose in 1931 when mathematician Kurt Gödel proved that any axiomatic system capable of describing arithmetic must contain true statements that cannot be proven within these axiomatic systems.

This means that no finite collections of rules can fully define infinity.

8 Taming Infinity

So how was infinity tamed?

Not by eliminating or ignoring it, but by changing how it is used.

Calculus replaced infinitesimals with limits.

Set theory replaced collections with axiomatic construction.

Logic used formal proof systems.

Mathematicians accepted that infinity behaves differently to finite numbers and stopped trying to force it to finite intuition.

Infinity became structured, disciplined, understood.

9 Conclusion

Infinity was once an abstract idea, something that is endlessly large but conceptually simple. However, when mathematicians tried to handle it, it created paradoxes, contradictions and crisis.

This led mathematics to grow up, to evolve.

The concept of infinity exposed weaknesses in reasoning and compelled the creation of deeper, stronger foundations.

Today, infinity has become a core part in calculus, topology, analysis and logic.

Infinity did not destroy maths, it transformed it…

10 References

[1] Zach, Richard, "Hilbert’s Program", *The Stanford Encyclopedia of Philosophy* (Winter 2023 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL = <https://plato.stanford.edu/archives/win2023/entries/hilbert-program/>.

[2] Huggett, Nick, "Zeno’s Paradoxes", *The Stanford Encyclopedia of Philosophy* (Winter 2025 Edition), Edward N. Zalta & Uri Nodelman (eds.), URL = <https://plato.stanford.edu/archives/win2025/entries/paradox-zeno/>.

[3] Hosch, William L.. "Cantor’s theorem". *Encyclopedia Britannica*, 15 Sep. 2016, https://www.britannica.com/science/Cantors-theorem. Accessed 20 February 2026.