Why Musical Scales Are Not Mathematically Perfect

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Introduction

From humanity's earliest perception of sound, mathematical ratios in pitch have always shaped what we perceive to be harmonious or musical. But why do certain patterns work?

In this essay I will investigate how musical scales and octave systems work mathematically and explore why having a perfect scale is fundamentally impossible. From these, I will try to build a musical system which is as close to perfection as mathematics allows.

Consonance and Dissonance

You might have heard of Pythagoras (Yes, **that** Pythagoras who discovered the formula $a^2 + b^2 = c^2$ which you learn at school). But did you know that he had a great influence on music as well?

Well, this comes from how maths and music are interconnected subjects. Let me ask you another question:

Have you ever wondered why certain pitch combinations sound particularly harmonious?

The answer is called Pythagorean tuning [1], where frequency ratios only involve powers of 2 and 3. Take a perfect fifth, one of the most consonant intervals for example: the notes C_4 and G_4 have frequencies of 261.63 Hz and 392.00 Hz respectively [2]. This would give a frequency ratio of approximately 2:3. In fact, notes which sound consonant would always have a simple frequency ratio: 4:3 for a perfect fourth, and 2:1 for an octave [3].

Now let's take two notes with a major second interval. You can understand this as the first two notes which are right next to each other in any major scale. For simplicity, I will take C_4 and D_4, which have frequencies of 261.63 Hz and 293.66 Hz respectively [2]:

$$\frac{293.66}{261.63} \approx 1.12$$

This would give a ratio of approximately 28:25. **Yikes**! If you played these two notes on the piano, it would sound as if they are "clashing". Musically, this is called dissonance and happens when frequency ratios aren't simple integers.

Stacking Fifths

Now, let us apply this mathematical knowledge to a core idea in music theory: the circle of fifths. Quite literally, this is a circle of notes, where each note is exactly a perfect fifth from its neighbouring notes. To visualise this, here's a diagram:

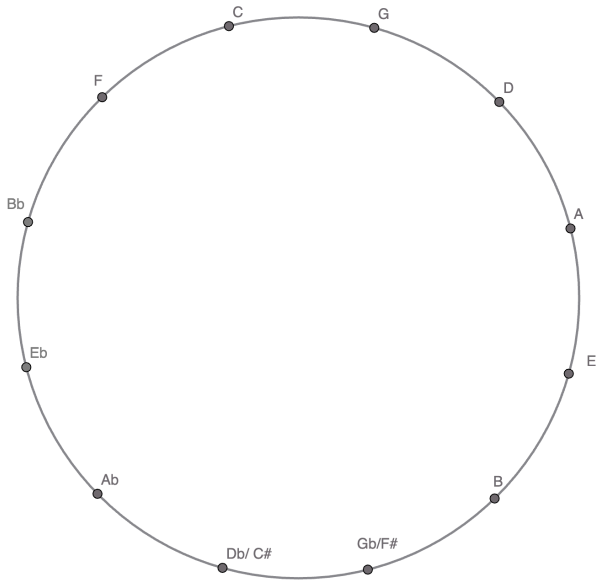

Figure 1: The Circle of Fifths

As you might have already noticed, if we move twelve steps around the circle, we will always end on the same note. According to this theory, surely if we stack 12 perfect fifths together, we will end on the same note from where we originally started seven octaves higher.

So, if the note is seven octaves higher, and the frequency doubles every octave, the ending note would be $2^7 = 128$ times the frequency of our original note. And now, using our stacking fifths theory, where the ratio is 3/2 for each perfect fifth, our frequency multiplier would be $\left(\frac{3}{2}\right)^{12} \approx 129.75$ times the frequency of our starting note.

But wait, I'm pretty sure 128 ≠ 129.75?

Well, dear reader, we have just discovered the Pythagorean comma [4]. This is the ratio between the two frequency multipliers:

$$\frac{\left(\frac{3}{2}\right)^{12}}{2^7} = \frac{3^{12}}{2^{19}} \approx 1.01.$$

This tiny gap is why no scales built from perfect fifths can be perfectly in tune. Some may wonder if another number other than 12 would close the gap – perhaps 13, or maybe 17? However, I will now prove why it would be impossible for any choice to close the gap:

Suppose integers *m* and *n* existed such that:

$$\left(\frac{3}{2}\right)^m = 2^n$$

Then:

$$\frac{3^m}{2^m} = 2^n$$

$$3^m = 2^{m+n}$$

However, 3^m is odd for any integer m, and 2^{m+n} is even for any integers $m + n$. Therefore, $3^m \neq 2^{m+n}$ and there is a contradiction.

This means that no matter what integers we choose for m and n, the Pythagorean comma would never close. This, however, raises an even deeper question: If musical scales cannot be perfectly in tune, what's the best we can do?

Mathematical Modelling

Using the Oxford Dictionary of Mathematics, the mathematical definition of a group is "a set closed under an operation, satisfying associativity, the existence of an identity, and inverses." [5].

In music, the twelve-note system forms a group under the operation of addition modulo 12, which could be denoted with:

Note	C	C#/Db	D	D#/Eb	E	F	F#/Gb	G	G#/Ab	A	A#/Bb	B
$\mathbb{Z}_{12}$	0	1	2	3	4	5	6	7	8	9	10	11

Group rules and how they are satisfied:

Rule	**Description** [6]	**How it works in a musical context**
Closure	The binary operation is closed: whenever $g, h \in G$, $gh \in G$	Adding any two notes would give another note e.g. $D + G = A$ (addition mod 12 so 2+7=9)
Associativity	The binary operation is associative: for any $g, h, k \in G$, $(gh)\,k = g\,(hk)$	The order of grouping does not affect the resulting note. e.g. $(C + E) + G = C + (E + G)$
Identity	There exists an identity element for the binary operation: an element $e \in G$ such that, for any g $\in G, eg = g = ge$	C is the identity. Adding C does not change the note (as in $\mathbb{Z}_{12}, C = 0$).
Inverse	Every element of has an inverse element with respect to the identity element named in the last rule: to each $g \in G$ there	Every note has an inverse. e.g. $C\# + B = C$

	corresponds $g^{-1} \in G$ such that $gg^{-1} = e = g^{-1}g$	

Remember the circle of fifths we just visited? For it to "tour" all twelve notes before returning to the start, the step size of 7 semitones must generate the entire group $\mathbb{Z}_{12}$. This means 7 and 12 are coprime as an element k generates $\mathbb{Z}_n$ **only** if $gcd(k, n) = 1$. In other words, $gcd(7,12) = 1$ [7].

Visualising Scales and Chords

The twelve tones of the scale could be visualised as twelve points which are equally spread apart on a circle:

Figure 2: Western Twelve Notes System

where each point is an element of $\mathbb{Z}_{12}$. A C Major/ A Natural Minor scale would be represented by the following polygon:

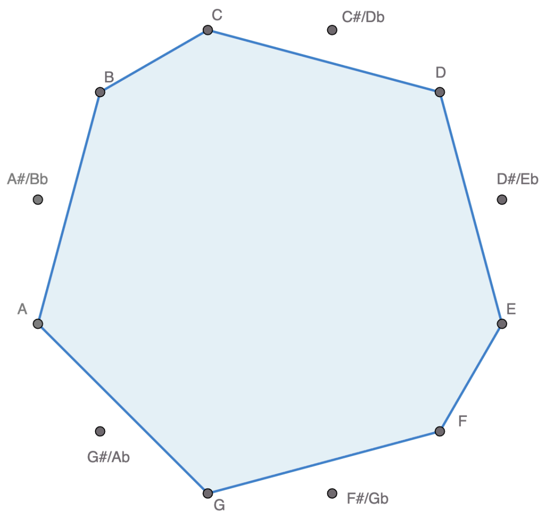

Figure 3: Polygon of C Major/ A Natural Minor Scale

Rotating the polygon by k steps applies the group operation $+k$ to every note simultaneously. This is formally known as a musical transposition, where the value of k determines the home note of the scale, for example: +2 results in a transposition to D Major/ B Natural Minor scale.

Now let's explore more scales (with a line of symmetry where applicable):

Harmonic Minor

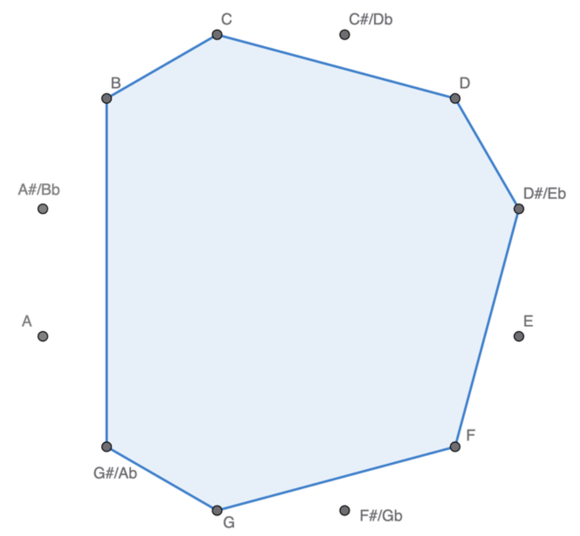

Figure 4: Polygon of C Harmonic Minor

Pentatonic Scale

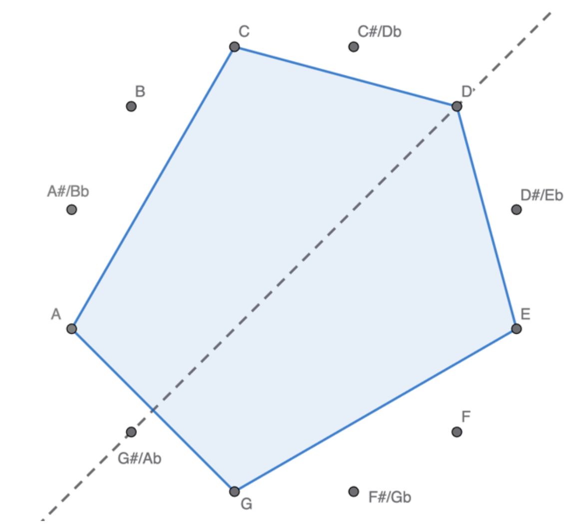

Figure 5: Polygon of C Pentatonic Scale

Whole Tone Scale

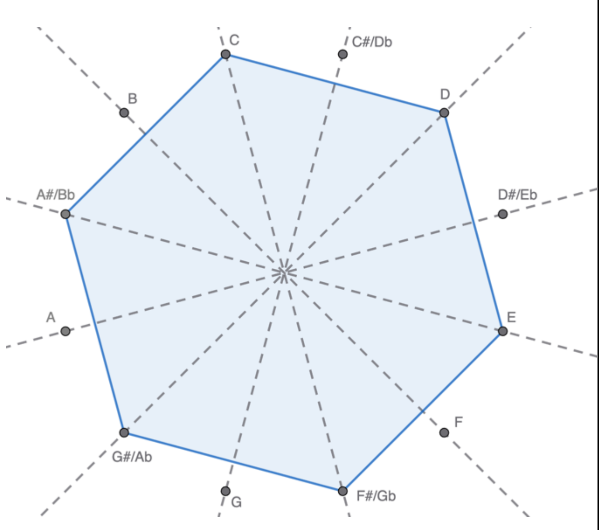

Figure 6: Polygon of Whole Tone Scale starting on C

Chromatic Scale

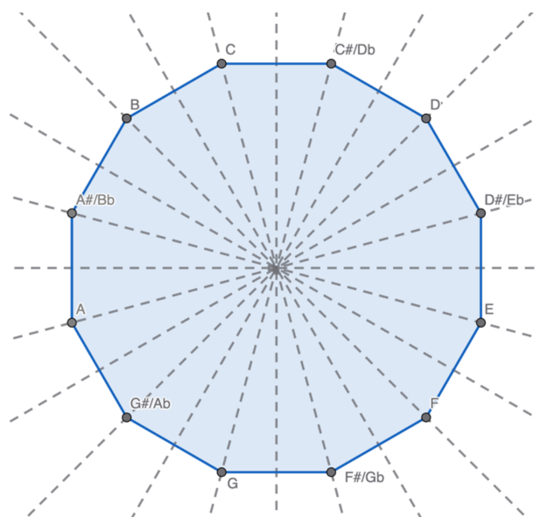

Figure 7: Polygon of Chromatic Scale

Interestingly, the scales which sound the most natural to our ears, the major and minor scales, are the least symmetrical. This is no coincidence: the fewer patterns there are in our scale, the harder it is to locate our home note. Perhaps this would explain why the chromatic scale, with the maximum 12-fold symmetry, is considered the most ambiguous scale musically [8].

Extending the model

We have looked at scales based on 12 notes, where (quite ironically,) there are imperfections in our "perfect fifths", where we understood that tuning is impossible to be perfect, where we understood symmetry means musical ambiguity. This raises our next question:

Is there a different number of notes we could use to minimise this imperfection?

In an *n*-tone equal temperament (n-TET) [9], an octave would be divided into *n*-equal steps, where each step would be $2^{\frac{1}{n}}$ times the frequency of the base note.

Suppose you take *k* steps for a perfect fifth, where *k* is any positive integer:

Then the frequency multiplier to *k* would be $2^{\frac{k}{n}}$.

Now, we equate this frequency multiplier to the ratio of a perfect fifth:

$$2^{\frac{k}{n}} = \frac{3}{2}$$

$$k = nlog_2(\frac{3}{2})$$

Keeping in mind that *k* must be an integer, we need to find a value of *n*, where there is minimal error between *k* and the perfect fifth:

$$\varepsilon(n) = |k - nlog_2\left(\frac{3}{2}\right)|$$

To find this value of *k* and *n*, we can try as many values as possible. In fact, let's look at the first 100 values of n:

n	*k*	Integer *k*	$\boldsymbol{\varepsilon(n)}$ (to 3s.f)
1	0.5849625	1	0.415
2	1.169925	1	0.17
3	1.7548875	2	0.245
4	2.33985	2	0.34
5	2.9248125	3	0.075
6	3.509775	4	0.49
7	4.09473751	4	0.095
8	4.67970001	5	0.32
9	5.26466251	5	0.265
10	5.84962501	6	0.15
11	6.43458751	6	0.435

n	*k*	Integer *k*	$\boldsymbol{\varepsilon(n)}$ (to 3s.f)
51	29.8330875	30	0.167
52	30.41805	30	0.418
53	31.0030125	31	0.003
54	31.587975	32	0.412
55	32.1729375	32	0.173
56	32.7579	33	0.242
57	33.3428625	33	0.343
58	33.927825	34	0.072
59	34.5127875	35	0.487
60	35.09775	35	0.098
61	35.6827125	36	0.317

12	7.01955001	7	0.02
13	7.60451251	8	0.395
14	8.18947501	8	0.189
15	8.77443751	9	0.226
16	9.35940001	9	0.359
17	9.94436251	10	0.056
18	10.529325	11	0.471
19	11.1142875	11	0.114
20	11.69925	12	0.301
21	12.2842125	12	0.284
22	12.869175	13	0.131
23	13.4541375	13	0.454
24	14.0391	14	0.039
25	14.6240625	15	0.376
26	15.209025	15	0.209
27	15.7939875	16	0.206
28	16.37895	16	0.379
29	16.9639125	17	0.036
30	17.548875	18	0.451
31	18.1338375	18	0.134
32	18.7188	19	0.281
33	19.3037625	19	0.304
34	19.888725	20	0.111
35	20.4736875	20	0.474
36	21.05865	21	0.059
37	21.6436125	22	0.356
38	22.228575	22	0.229
39	22.8135375	23	0.186
40	23.3985	23	0.399
41	23.9834625	24	0.017
42	24.568425	25	0.432
43	25.1533875	25	0.153
44	25.73835	26	0.262
45	26.3233125	26	0.323
46	26.908275	27	0.092
47	27.4932375	27	0.493
48	28.0782	28	0.078
49	28.6631625	29	0.337
50	29.248125	29	0.248

62	36.267675	36	0.268
63	36.8526375	37	0.147
64	37.4376	37	0.438
65	38.0225625	38	0.023
66	38.607525	39	0.392
67	39.1924875	39	0.192
68	39.77745	40	0.223
69	40.3624125	40	0.362
70	40.9473751	41	0.053
71	41.5323376	42	0.468
72	42.1173001	42	0.117
73	42.7022626	43	0.298
74	43.2872251	43	0.287
75	43.8721876	44	0.128
76	44.4571501	44	0.457
77	45.0421126	45	0.042
78	45.6270751	46	0.373
79	46.2120376	46	0.212
80	46.7970001	47	0.203
81	47.3819626	47	0.382
82	47.9669251	48	0.033
83	48.5518876	49	0.448
84	49.1368501	49	0.137
85	49.7218126	50	0.278
86	50.3067751	50	0.307
87	50.8917376	51	0.108
88	51.4767001	51	0.477
89	52.0616626	52	0.062
90	52.6466251	53	0.353
91	53.2315876	53	0.232
92	53.8165501	54	0.183
93	54.4015126	54	0.402
94	54.9864751	55	0.014
95	55.5714376	56	0.429
96	56.1564001	56	0.156
97	56.7413626	57	0.259
98	57.3263251	57	0.326
99	57.9112876	58	0.089
100	58.4962501	58	0.496

Just in case, for your visualisation, here's a simple scatter graph:

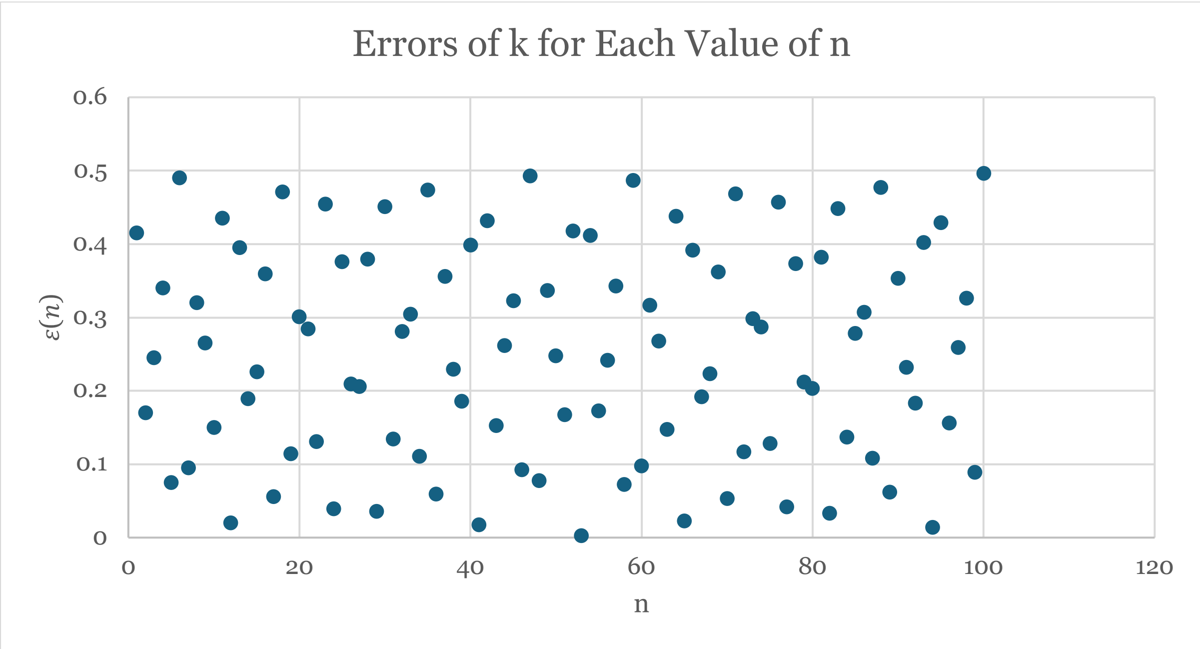

Figure 8: Table of errors of k for each value of n

And if this looks like just any no-correlation-scatter graph, don't worry – we are not looking for any correlation of just any n. I have re-plotted our graph using GeoGebra, along with a horizontal line at $y = 0.02$ (the error for 12-TET), so that we know that the error values that fall under the line might be a better solution than 12-TET.

Here are our fascinating results:

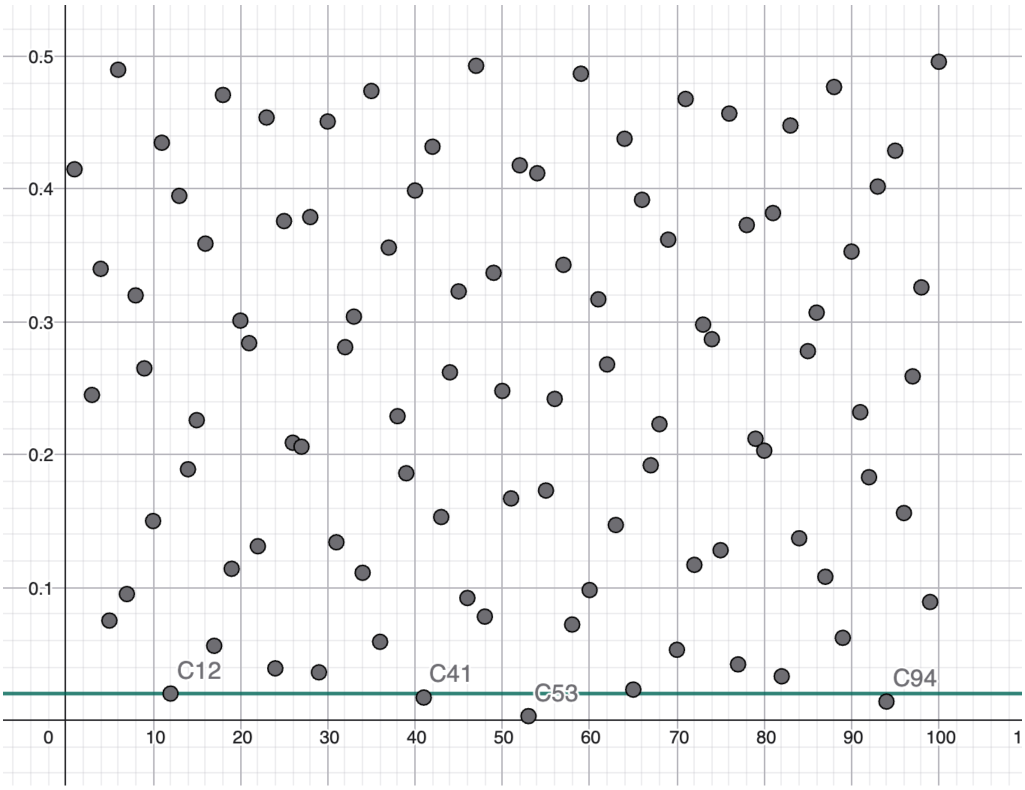

Figure 9: Plotted graph of k and n values on GeoGebra

The next best value after the original 12 notes per octave would be a 41-TET, followed by 53-TET and 94-TET.

The mathematically optimised scale

Well, that was a lot – I'm glad spreadsheet software exists. But there's got to be a better approach to this problem, right?

Let's rewind back to where we got our formula:

$$k = nlog_2(\frac{3}{2})$$

Rearranging the equation would give:

$$\frac{k}{n} = \log_2(\frac{3}{2})$$

Now, we can represent $\log_2 \frac{3}{2}$ as a continued fraction to give rational approximations for $\frac{k}{n}$. [10] Let c_i be the convergent of the continued fraction $[a_0;\ a_1, a_2, a_3 \dots a_n]$ at stage *i,* and let $c_i = \frac{k_i}{n_i}$. I will also use b_i to represent the remainder at each stage.

Recurrence relations: $k_i = a_i k_{i-1} + k_{i-2}$ and $n_i = a_i n_{i-1} + n_{i-2}$

In $\log_2 \frac{3}{2}$:

$$a_0 = 0 \text{ as } \log_2\left(\frac{3}{2}\right) < 1$$

Let $b_1 = \frac{3}{2}$,

$$\left(\frac{3}{2}\right)^1 < 2 < \left(\frac{3}{2}\right)^2 \Rightarrow a_1 = 1$$

$$b_2 = \frac{2}{\left(\frac{3}{2}\right)^1} = \frac{2^2}{3}$$

b2:

$$\left(\frac{2^2}{3}\right)^1 < \frac{3}{2} < \left(\frac{2^2}{3}\right)^2 \Rightarrow a_2 = 1$$

$$b_3 = \frac{\left(\frac{3}{2}\right)}{\left(\frac{2^2}{3}\right)^1} = \frac{3^2}{2^3}$$

b_3:

$$\left(\frac{3^2}{2^3}\right)^2 < \frac{2^2}{3} < \left(\frac{3^2}{2^3}\right)^3 \Rightarrow a_3 = 2$$

$$b_4 = \frac{(\frac{2^2}{3})}{\left(\frac{3^2}{2^3}\right)^2} = \frac{2^8}{3^5}$$

I then continued this working until I got the value of a_7, which gets us to:

$$[0; 1,1,2,2,3,1,5 \ldots a_n]$$

This means the continued fraction is:

$$c_i = 0 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{2 + \cfrac{1}{3 + \cfrac{1}{1 + \cfrac{1}{5 + \cdots}}}}}}}$$

In fact, here's a table to put in context exactly what we are doing:

i	b_i	a_i	c_i	k_i	n_i
0	-	0	$\frac{0}{1}$	0	1
1	$\frac{3}{2}$	1	$\frac{1}{1}$	1	1
2	$\frac{2^2}{3}$	1	$\frac{1}{2}$	1	2
3	$\frac{3^2}{2^3}$	2	$\frac{3}{5}$	3	5
4	$\frac{2^8}{3^5}$	2	$\frac{7}{12}$	7	12
5	$\frac{3^{12}}{2^{19}}$	3	$\frac{24}{41}$	24	41
6	$\frac{2^{65}}{3^{41}}$	1	$\frac{31}{53}$	31	53
7	$\frac{3^{53}}{2^{84}}$	5	$\frac{179}{306}$	179	306

As you can see on the table, we have now obtained values for n, the number of notes in a scale for minimal error. To you who might be checking this with our previously found values, using a continued fraction like this gives values of n whose error gets progressively smaller. This is the reason why n_7 is not 94 as we would have expected.

Discussion

After exploring the properties of musical scales, let us finally come to a stop and ask our final question:

What exactly makes a perfect scale?

Based on the explorations we have done, I think we can unanimously agree that a perfect scale should have the following criteria:

1. Have good tuning accuracy i.e. the value of $\varepsilon(n)$ must be small
2. Have a value of k which must generate $\mathbb{Z}_n$ i.e. $\gcd(k, n) = 1$
3. Have minimal symmetry so there is a tonal direction
4. (For the sake of musicians,) our system of notes must be as small as possible

From the previous section, we have found out that the musical systems with a tuning accuracy of the current 12-note scale or better would be 12-TET, 41-TET, 53-TET, 306-TET. Of course, if we continued our convergent values, the value of n could be infinitely larger (literally!). However, realistically, having so many notes just for a scale would be quite impractical and not functional.

Unsurprisingly, these values of n (12, 41, 53, 306) would also satisfy our second criterion, as convergents of continued fractions are always in their lowest terms. This means that, conveniently, each of these systems would already have their own working "circle of fifths" as their value of gcd (k, n) would always equal 1.

For the last criterion, we will be using Lagrange's theorem in group theory, which states that: "if G is a finite group of order n, and H is a subgroup of G, then the order of $|H|$ divides the order of $|G|$." [11] This means that in our four numbers, having a scale within 41-TET or 53-TET would always guarantee a unique tonal centre, as 41 and 53 are both prime numbers with factors of 1 and itself only.

Conclusion

Before I answer our last question, I would like to thank you, the reader, for joining me in this mathematically musical journey. Who would have thought that even the basics of music are so mathematical!

At the end, it truly depends on your interpretation of what is closest to "perfect". If we prioritise practicality (and of course, our fellow musicians' sanity), 12-TET would be the best solution, giving us the smallest system with acceptable tuning accuracy. Yet, if we want to prioritise a tonal centre, we might want to consider 41-TET, being the smallest prime system. However, if all we want is tonal accuracy, it might be worth looking at 53-TET, as it has good accuracy and a unique tonal centre. Perhaps you might even want more accuracy – continuing our convergents from the continued fractions.

Indeed, musical scales are not perfect. Mathematics does not allow us to compose the perfect scale: it tells us exactly why scales cannot be perfect, and precisely how close we can come to our interpretation of the "perfect scale".

References

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(Accessed: 12 April 2026).

Tools Used

All calculations were performed on a CASIO fx-CG50 calculator.
Large calculations of data sets and simple graphs were plotted on Microsoft Excel.
Polygons and complex graphs were plotted and drawn on GeoGebra.